On the homotopy category of A_∞-categories

Master's Thesis

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Chapter 0

Introduction

0.1 Motivation

0.1.1 A_∞-algebras

An A_∞-algebra is a $\mathbf{Z}$-graded module A together with maps $\mathsf{m}_k \colon A^{\otimes k} \to A$ of degree $2-k$ for $k \geq 1$ that satisfy generalised associativity relations. In particular, one has $\mathsf{m}_1\mathsf{m}_1 = 0$, i.e. m_1 is a differential. Thus complexes are special cases of A_∞-algebras with $\mathsf{m}_k = 0$ for $k \geq 2$. Another special case are differential graded algebras, which are A_∞-algebras with $\mathsf{m}_k = 0$ for $k \geq 3$.

0.1.2 A_∞-categories

One can generalise A_∞-algebras to A_∞-categories, just as monoids can be generalised to categories. For instance, given morphisms $a_1 \colon x_0 \to x_1$, $a_2 \colon x_1 \to x_2$ and $a_3 \colon x_2 \to x_3$, we obtain a morphism $(a_1 \otimes a_2 \otimes a_3)\mathsf{m}_3$ from x_0 to x_3. Again, the maps m_k for $k \geq 1$ are required to satisfy generalised associativity relations.

0.1.3 A_∞-categories preserve cohomological information

Let B be an algebra over a field and let $M_1, \dots, M_n$ be B-modules. For each i we choose a projective resolution P_i of M_i. Then we can define a differential graded category with objects given by the numbers $1, \dots, n$ and with $\mathrm{Hom}(i,j)$ given by the complex of graded linear maps $P_i \to P_j$ of arbitrary degree with differential

$$f\delta := f d_{P_j} - (-1)^p d_{P_i} f$$

for a graded linear map $f \colon P_i \to P_j$ of degree p.

By a theorem of Kadeishvili there exists a minimal model for this differential graded category. This minimal model is an A_∞-category that has also the numbers $1, \dots, n$ as objects, but it has $\mathrm{Hom}(i,j) = \mathrm{Ext}^*_B(M_i, M_j)$ with zero differential. There is an A_∞-quasiisomorphism from the minimal model to the original differential graded category. In this situation, the minimal model is unique up to A_∞-isomorphism.

Our minimal model $(\mathrm{Ext}^*_B(M_i, M_j))_{i,j}$ has the Yoneda product as multiplication map m_2. In general, the higher multiplication maps m_k for $k \geq 3$ are non-zero, i.e. the minimal model is not a differential graded category.

One can recover the full subcategory of B-Mod consisting of those B-modules that have a filtation with all subfactors in $\{M_1, \ldots, M_n\}$ from the A_∞-category via the filt-construction, cf. [Kel01, §7.7] and [Lef03, §7.4].

If we generalise from a ground field to a commutative ground ring, not every differential graded category has a minimal model in the sense described above. In [Sag10] and [Sch15] versions of A_∞-categories over a commutative ground ring are considered that allow minimal models in a suitable sense.

0.2 Problems

In what follows, we consider a commutative ground ring R.

0.2.1 The grading formalism

We introduce the notion of a grading category and graded modules over a grading category, cf. Definitions 3 and 6. A grading category is a category $\mathcal{Z}$ with additional data. A $\mathcal{Z}$-graded module is a tuple $M = (M^z)_{z \in \mathrm{Mor}(\mathcal{Z})}$ of modules M^z.

For instance, we may let $\mathcal{Z} = \mathbf{Z}$, where the integers $\mathbf{Z}$ are regarded as a category with one object and morphisms $\mathrm{Mor}(\mathbf{Z}) = \mathbf{Z}$ with addition as composition. This gives $\mathbf{Z}$-graded modules in the classical sense. An A_∞-algebra over $\mathbf{Z}$ is an A_∞-algebra in the classical sense.

But we may also let $\mathcal{Z} = \mathbf{Z} \times \mathsf{Pair}(X)$, where $\mathsf{Pair}(X)$ is the pair category over a set X, cf. Definition 5. Then an A_∞-algebra over $\mathcal{Z}$ is an A_∞-category with set of objects X.

In what follows, we fix a grading category $\mathcal{Z}$. Unless stated otherwise, graded means $\mathcal{Z}$-graded. To a differential graded module we shall also refer as a complex.

0.2.2 The Bar construction

Consider the categories A_∞-alg of A_∞-algebras and $\mathsf{dgCoalg}$ of differential graded coalgebras. The Bar functor is a full and faithful functor

$$\mathrm{Bar}\colon \quad \mathrm{A}_\infty\text{-}\mathsf{alg} \longrightarrow \mathsf{dgCoalg}\,.$$

Given an A_∞-algebra A, the differential graded coalgebra $\mathrm{Bar}\,A$ is a tensor coalgebra $TA^{[1]}$ with a differential that depends on the multiplication maps on A.

So the image of Bar is the category $\mathsf{dtCoalg}$ of differential graded coalgebras whose underlying graded coalgebra is a tensor coalgebra, called differential graded tensor coalgebras, cf. §1.3.3. Thus the category A_∞-alg is equivalent to the category $\mathsf{dtCoalg}$.

0.2.3 The aim

We want to construct and study the homotopy category of A_∞-algebras. That is, we want to define a notion of homotopy, i.e. a congruence relation on the category A_∞-alg. As complexes are special cases of A_∞-algebras, this homotopy notion should have the usual notion of complex homotopy as a special case.

Morphisms of A_∞-algebras are tuples $(f_k)_{k\geq 1}$ of graded linear maps satisfying certain equations. In particular, the component f_1 is a complex morphism, i.e. $f_1 \mathsf{m}_1 = \mathsf{m}_1 f_1$. Prouté's theorem states that over a ground field a morphism of A_∞-algebras is an A_∞-homotopy equivalence if and only if f_1 is a quasiisomorphism of complexes, cf. [Pro84, Théorème 4.27], see also [Kel01, Theorem in §3.7] and [Sei08, Corollary 1.14].

The naive generalisation to a commutative ground ring R fails, as quasiisomorphisms of complexes of R-modules do not need to be homotopy equivalences of complexes. We want to give a suitable generalisation of Prouté's theorem that characterises homotopy equivalences over a commutative ground ring.

0.3 Results

0.3.1 An A_∞-category of coderivations

Let A and B be graded modules. Consider the tensor coalgebras TA and TB. Write Δ for the respective comultiplication. Suppose given differentials such that TA and TB form differential graded coalgebras. Then TA and TB are objects in $\mathsf{dtCoalg}$, i.e. differential graded tensor coalgebras.

For morphisms of differential graded coalgebras $f, g \colon TA \to TB$ we define the notion of an (f, g)-coderivation, cf. Definition 34. Such an (f, g)-coderivation is a graded linear map $h \colon TA \to TB$ of some degree that satisfies

$$h\Delta = \Delta(f \otimes h + h \otimes g).$$

Let $\mathsf{dgCoalg}(TA, TB)$ denote the set of morphisms of differential graded coalgebras between TA and TB. Consider the grading category $\mathcal{Z}_{TA,TB} := \mathbf{Z} \times \mathsf{Pair}(\mathsf{dgCoalg}(TA, TB))$. Let $\mathrm{Coder}(TA, TB)$ be the $\mathcal{Z}_{TA,TB}$-graded module such that $\mathrm{Coder}(TA, TB)^{p,(f,g)}$ is the module of (f, g)-coderivations of degree p for $(p, (f, g)) \in \mathrm{Mor}(\mathcal{Z}_{TA,TB})$.

The following theorem is our version of various theorems in the literature, established by Fukaya [Fuk02, Theorem-Definition 7.55], Seidel [Sei08, §1d], Lefèvre-Hasegawa [Lef03, Lemme 8.1.1.4] and Lyubashenko [Lyu03, Proposition 5.1] in various degrees of generality.

Theorem 49 *There is a structure of an A_∞-algebra on $\mathrm{Coder}(TA, TB)$ such that the corresponding differential M on $T\,\mathrm{Coder}(TA, TB)$ fits into a certain commutative square.*

One can interpret the A_∞-algebra $\mathrm{Coder}(TA, TB)$ as an A_∞-category with objects given by morphisms of differential graded coalgebras and morphisms given by coderivations between them.

This A_∞-structure has been constructed by Fukaya, Seidel and Lefèvre-Hasegawa in the case of R being a field and without making use of the Bar construction. Lyubashenko translates it to the context of $\mathsf{dtCoalg}$, which simplifies the resulting formulas. We characterise them via the mentioned commutative square.

0.3.2 Construction of the homotopy category

Let TA and TB be differential graded tensor coalgebras. Let $f, g\colon TA \to TB$ be morphisms of differential graded coalgebras.

A coderivation homotopy from f to g is an (f, g)-coderivation $h\colon TA \to TB$ of degree -1 that satisfies $f - g = hm_{TA} + m_{TB}h$, where m_{TA} and m_{TB} denote the differentials on TA and TB respectively. The morphisms f and g are called coderivation homotopic if there is a coderivation homotopy from f to g.

Theorem 63 *Being coderivation homotopic is a congruence on* $\mathsf{dtCoalg}$.
Via the Bar construction, it also defines a congruence on the category $\mathrm{A}_\infty\text{-}\mathsf{alg}$ *of* A_∞*-algebras. We obtain the equivalent factor categories* $\underline{\mathsf{dtCoalg}}$ *and* $\mathrm{A}_\infty\text{-}\underline{\mathsf{alg}}$.

Note that if h is a homotopy from f to g, then $-h$ is in general not a homotopy from g to f, as it may not be a (g, f)-coderivation. Similarly, if h' is a homotopy from f to f' and h'' a homotopy from f' to f'', then $h' + h''$ is in general not an (f, f'')-coderivation and thus not a homotopy from f to f''. In both cases, correction terms are needed.

To prove this theorem, we essentially translate the arguments in Seidel's book, cf. [Sei08, §1h], to our context. More precisely, we work over a commutative ground ring and give explicit formulas for all construction on the differential graded coalgebra side of the Bar construction. The A_∞-category of coderivations is used in the proof to produce the required correction terms.

0.3.3 A generalisation of a theorem of Prouté

A morphism of A_∞-algebras f in $\mathrm{A}_\infty\text{-}\mathsf{alg}$ is called an A_∞-homotopy equivalence if its residue class $[f]$ is an isomorphism in $\mathrm{A}_\infty\text{-}\underline{\mathsf{alg}}$.

Theorem 79 *A morphism of* A_∞*-algebras* f *is an* A_∞*-homotopy equivalence if and only if its first component* f_1 *is a homotopy equivalence of complexes.*

Over a ground field, quasiisomorphisms of complexes are precisely the homotopy equivalences of complexes. Hence this theorem generalises Prouté's theorem.

In fact, we have a functor $V\colon \mathsf{dtCoalg} \to \mathsf{dgMod}$ from the category of differential graded tensor coalgebras to the category of differential graded modules, i.e. complexes, mapping $(f\colon TA \to TB) \mapsto (f|_A^B\colon A \to B)$. The functor V induces a functor $\bar{V}$ between the respective homotopy categories, cf. Lemma 68. We obtain the following commutative diagram of functors, where the vertical functors are the residue class functors.

$$\begin{array}{ccccc}
\mathrm{A}_\infty\text{-}\mathsf{alg} & \xrightarrow[\sim]{\mathrm{Bar}} & \mathsf{dtCoalg} & \xrightarrow{V} & \mathsf{dgMod} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{A}_\infty\text{-}\underline{\mathsf{alg}} & \xrightarrow[\sim]{} & \underline{\mathsf{dtCoalg}} & \xrightarrow{\bar{V}} & \underline{\mathsf{dgMod}}
\end{array}$$

The above theorem states that $\bar{V}$ reflects isomorphisms.

We give examples that show that $\bar{V}$ is in general neither full nor faithful, cf. Remark 81.

0.3.4 The homotopy category as a localisation

We show that two coderivation homotopic maps in dtCoalg fit into a certain commutative diagram involving coderivation homotopy equivalences. We use this diagram to show that any functor dtCoalg $\to \mathcal{D}$ that maps homotopy equivalences to isomorphisms has to map two coderivation homotopic maps to the same morphism. Hence we obtain the following theorem.

Theorem 92 *The category* $\underline{\text{dtCoalg}}$ *is the localisation of* dtCoalg *at the set of coderivation homotopy equivalences.*
Using the Bar construction, it follows that A_∞-$\underline{\text{alg}}$ *is the localisation of* A_∞-alg *at the set of* A_∞*-homotopy equivalences.*

0.4 Relations to work of Lefèvre-Hasegawa

Lefèvre-Hasegawa constructs in his thesis [Lef03] a model structure on a full subcategory of certain differential graded coalgebras over a ground field. The construction is based on work of Hinich, cf. [Hin97]. The bifibrant objects of this model structure turn out to be the differential graded tensor coalgebras, i.e. the objects dtCoalg.

He then shows that the homotopy notion of this model structure coincides with the one given by coderivation homotopy, which proves that coderivation homotopy is a congruence. Moreover, the weak equivalences of this model structure are the A_∞-quasiisomorphisms, hence Prouté's theorem and the theorem on localisation above also follow from Lefèvre's model structure over a ground field.

In the proof of our generalisation of Prouté's theorem, cf. §3.2, we make use of arguments inspired by Lefèvre's work without actually constructing a full model structure. In particular, we translate some of Lefèvre's lemmas to our context, but reprove them to show that they also hold over a commutative ground ring.

To construct a full model structure that has dtCoalg as bifibrant objects, one would have to introduce a subcategory dtCoalg $\subseteq \mathcal{X} \subseteq$ dgCoalg that would presumably require a rather technical definition. It is more convenient to only consider dtCoalg.

0.5 Conventions

Sets and functions

- Composition of morphisms is written on the right, i.e. the composite of $f\colon X \to Y$ and $g\colon Y \to Z$ is denoted by $fg\colon X \to Z$.
- If $f\colon X \to Y$ is a map between sets, we write xf for the image of $x \in X$ under f.
- We write $\mathbf{Z}$ for the ring of integers.

Categories and functors

- Given a category $\mathcal{C}$, we write $\mathrm{Ob}(\mathcal{C})$ for the set of objects and $\mathrm{Mor}(\mathcal{C})$ for the set of morphisms of $\mathcal{C}$.

- The opposite category of $\mathcal{C}$ is denoted by $\mathcal{C}^{\mathrm{op}}$.
- We write $\mathrm{id}_X \colon X \to X$ for the identity morphisms on an object $X \in \mathrm{Ob}(\mathcal{C})$ in a category $\mathcal{C}$. We often omit the index and write $\mathrm{id} := \mathrm{id}_X$.
- Given a category $\mathcal{C}$ and two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, we write $\mathcal{C}(X, Y)$ for the set of morphisms from X to Y.
- A functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$ is also called a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$.
- Composition of functors is written on the left, i.e. the composite of $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{E}$ is denoted by $G \circ F \colon \mathcal{C} \to \mathcal{E}$.
- Given a functor $F \colon \mathcal{C} \to \mathcal{D}$ and a morphism $f \colon X \to Y$ in $\mathcal{C}$, we write $Ff \colon FX \to FY$ for its image under F in $\mathcal{D}$.

Modules and linear maps

- All modules are left modules over a commutative ring R. Given $r \in R$ and $m \in M$, we also write $mr := rm$, i.e. we consider left modules as right modules with the same R-operation.
- We usually fix a commutative ring R and write module for R-module and linear map for R-linear map. Moreover, tensor products are always considered as tensor products over the ground ring R.
- Given two modules M and N, we write $\mathrm{Hom}(M, N)$ for the set of linear maps from M to N.

Graded modules and graded linear maps (see also §1.2)

Let $\mathcal{Z}$ be a grading category, see Definition 3 below.

- Suppose given a $\mathcal{Z}$-graded linear map $f \colon M \to N$ of degree $p \in \mathbf{Z}$ and $z \in \mathrm{Mor}(\mathcal{Z})$. Given $m \in M^z$, we often write $mf := mf^z \in N^{z[p]}$, i.e. we omit the degree on f.
- A $\mathcal{Z}$-graded linear map $f \colon M \to N$ of degree $p \in \mathbf{Z}$ is called injective, surjective resp. bijective, if $f^z \colon M^z \to N^{z[p]}$ is an injective, surjective or bijective linear map for all $z \in \mathrm{Mor}(\mathcal{Z})$.
- We write $\mathsf{grHom}(M, N)$ for the set of $\mathcal{Z}$-graded linear maps between the $\mathcal{Z}$-graded modules M and N.

Chapter 1

Preliminaries

1.1 Adjunctions

Let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ be a pair of functors F and G between categories $\mathcal{C}$ and $\mathcal{D}$.

We recall the property of adjointness with its equivalent characterisations by a natural isomorphism between hom-sets, unit and counit and a natural transformation with a universal property.

Definition 1 We call F *left adjoint* to G (or G *right adjoint* to F) if there is a natural isomorphism

$$\varphi\colon \mathcal{C}(-, G(=)) \xrightarrow{\sim} \mathcal{D}(F(-), =)$$

in the category of functors from $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ to the category of sets.

We write $F \dashv G$ and say that (F, G) is an *adjoint pair*.

Lemma 2 (cf. [Mac98, Theorem 2, p. 93]) *The following are equivalent.*

(1) *The functor F is left adjoint to G, i.e. $F \dashv G$.*

(2) *There are natural transformations $\eta\colon \mathrm{id}_{\mathcal{C}} \to GF$ and $\varepsilon\colon FG \to \mathrm{id}_{\mathcal{D}}$ such that the following diagrams commute for all $X \in \mathrm{Ob}(\mathcal{C})$ and $Y \in \mathrm{Ob}(\mathcal{D})$.*

$$\begin{array}{ccc} FX & \xrightarrow{F\eta_X} & FGFX \\ & {}_{\mathrm{id}_{FX}}\searrow & \downarrow{\scriptstyle \varepsilon_{FX}} \\ & & FX \end{array} \qquad\qquad \begin{array}{ccc} GY & \xrightarrow{\eta_{GY}} & GFGY \\ & {}_{\mathrm{id}_{GY}}\searrow & \downarrow{\scriptstyle G\varepsilon_Y} \\ & & GY \end{array}$$

(3) *There is a natural transformation $\varepsilon\colon FG \to \mathrm{id}_{\mathcal{D}}$ and for each morphism $f\colon FX \to Y$ in $\mathcal{D}$ there is a unique morphism $\bar{f}\colon X \to GY$ such that $f = (F\bar{f})\varepsilon_Y$.*

$$\begin{array}{ccc} FX & \xrightarrow{f} & Y \\ & {}_{\exists! F\bar{f}}\searrow & \uparrow{\scriptstyle \varepsilon_Y} \\ & & FGY \end{array}$$

If $F \dashv G$ is an adjoint pair of functors, the natural transformation $\varepsilon\colon FG \to \mathrm{id}_{\mathcal{D}}$ from Lemma 2.(2) is called a *counit* while $\eta\colon \mathrm{id}_{\mathcal{C}} \to GF$ is called a *unit* of the adjunction.

Proof. (1) $\Rightarrow$ (2) For objects $X \in \mathrm{Ob}(\mathcal{C})$ and $Y \in \mathrm{Ob}(\mathcal{D})$ define morphisms $\eta_X\colon X \to GFX$ and $\varepsilon_Y\colon FGY \to Y$ by

$$\eta_X := (\mathrm{id}_{FX})\varphi^{-1}_{X,FX} \qquad \text{and} \qquad \varepsilon_Y := (\mathrm{id}_{GY})\varphi_{GY,Y}.$$

Note that since φ is a natural isomorphism also $\varphi^{-1}\colon \mathcal{D}(F(-),=) \to \mathcal{C}(-,G(=))$ is a natural isomorphism with components $\left(\varphi^{-1}\right)_{X,Y} := \varphi^{-1}_{X,Y}$.

Suppose given a morphism $f\colon X' \to X$ in $\mathcal{C}$. Then using the naturality of φ^{-1} we have

$$\begin{aligned} f\eta_X = f\cdot(\mathrm{id}_{FX})\varphi^{-1}_{X,FX} &= (\mathrm{id}_{FX})\varphi^{-1}_{X,FX}\mathcal{C}(f, G\,\mathrm{id}_{FX}) \\ &= (\mathrm{id}_{FX})\mathcal{D}(Ff, \mathrm{id}_{FX})\varphi^{-1}_{X',FX} = (Ff)\varphi^{-1}_{X',FX}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \eta_{X'}(GFf) = (\mathrm{id}_{FX'})\varphi^{-1}_{X',FX'}\cdot(GFf) &= (\mathrm{id}_{FX'})\varphi^{-1}_{X',FX'}\mathcal{C}(\mathrm{id}_{X'}, G(Ff)) \\ &= (\mathrm{id}_{FX'})\mathcal{D}(F\,\mathrm{id}_{X'}, Ff)\varphi^{-1}_{X',FX} = (Ff)\varphi^{-1}_{X',FX}. \end{aligned}$$

We conclude that $\eta := (\eta_X)_{X\in\mathrm{Ob}(\mathcal{C})}$ constitutes a natural transformation $\eta\colon \mathrm{id}_{\mathcal{C}} \to GF$.

Suppose given a morphism $g\colon Y \to Y'$ in $\mathcal{D}$. Then using the naturality of φ we have

$$\begin{aligned} \varepsilon_Y g = (\mathrm{id}_{GY})\varphi_{GY,Y}\cdot g &= (\mathrm{id}_{GY})\varphi_{GY,Y}\mathcal{D}(F(\mathrm{id}_{GY}), g) \\ &= (\mathrm{id}_{GY})\mathcal{C}(\mathrm{id}_{GY}, Gg)\varphi_{GY,Y'} = (Gg)\varphi_{GY,Y'}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} (FGg)\varepsilon_{Y'} = (FGg)\cdot(\mathrm{id}_{GY'})\varphi_{GY',Y'} &= (\mathrm{id}_{GY'})\varphi_{GY',Y'}\mathcal{D}(F(Gg), \mathrm{id}_{Y'}) \\ &= (\mathrm{id}_{GY})\mathcal{C}(Gg, G(\mathrm{id}_{Y'}))\varphi_{GY,Y'} = (Gg)\varphi_{GY,Y'}. \end{aligned}$$

We conclude that $\varepsilon := (\varepsilon_Y)_{Y\in\mathrm{Ob}(\mathcal{D})}$ constitutes a natural transformation $\varepsilon\colon FG \to \mathrm{id}_{\mathcal{D}}$.

For the first asserted commutative triangle we calculate using the naturality of φ for $X \in \mathrm{Ob}(\mathcal{C})$

$$\begin{aligned} (F\eta_X)(\varepsilon_{FX}) &= F\big((\mathrm{id}_{FX})\varphi^{-1}_{X,FX}\big)\cdot(\mathrm{id}_{GFX})\varphi_{GFX,FX} \\ &= (\mathrm{id}_{GFX})\varphi_{GFX,FX}\mathcal{D}\big(F\big((\mathrm{id}_{FX})\varphi^{-1}_{X,FX}\big), \mathrm{id}_{FX}\big) \\ &= (\mathrm{id}_{GFX})\mathcal{C}\big((\mathrm{id}_{FX})\varphi^{-1}_{X,FX}, G(\mathrm{id}_{FX})\big)\varphi_{X,FX} \\ &= (\mathrm{id}_{FX})\varphi^{-1}_{X,FX}\varphi_{X,FX} \\ &= \mathrm{id}_{FX}\,. \end{aligned}$$

For the second asserted commutative triangle we also use naturality of φ^{-1} for $Y \in \mathrm{Ob}(\mathcal{D})$ and obtain

$$\begin{aligned} (\eta_{GY})(G\varepsilon_Y) &= (\mathrm{id}_{FGY})\varphi^{-1}_{GY,FGY}\cdot G\big((\mathrm{id}_{GY})\varphi_{GY,Y}\big) \\ &= (\mathrm{id}_{FGY})\varphi^{-1}_{GY,FGY}\mathcal{C}\big(\mathrm{id}_{GY}, G\big((\mathrm{id}_{GY})\varphi_{GY,Y}\big)\big) \\ &= (\mathrm{id}_{FGY})\mathcal{D}\big(F(\mathrm{id}_{GY}), (\mathrm{id}_{GY})\varphi_{GY,Y}\big)\varphi^{-1}_{GY,Y} \\ &= (\mathrm{id}_{GY})\varphi_{GY,Y}\varphi^{-1}_{GY,Y} \\ &= \mathrm{id}_{GY}\,. \end{aligned}$$

(2) ⇒ (3) By assumption, there is a natural transformation $\varepsilon\colon FG \to \mathrm{id}_{\mathcal{D}}$. Suppose given a morphism $f\colon FX \to Y$ in $\mathcal{D}$. Consider $\bar{f} := \eta_X(Gf)\colon X \to GY$. Then using naturality of ε and the first commutative triangle in the assumptions we obtain

$$(F\bar{f})\varepsilon_Y = (F\eta_X)(FGf)\varepsilon_Y = (F\eta_X)\varepsilon_{FX} f = f.$$

To show uniqueness, suppose given morphisms $\bar{f}_1\colon X \to GY$ and $\bar{f}_2\colon X \to GY$ in $\mathcal{C}$ with $f = (F\bar{f}_1)\varepsilon_Y = (F\bar{f}_2)\varepsilon_Y$. Applying G to this equation and precomposing with η_X gives

$$\eta_X(GF\bar{f}_1)(G\varepsilon_Y) = \eta_X(GF\bar{f}_2)(G\varepsilon_Y).$$

Now use naturality of η and the second commutative triangle in the assumptions to obtain

$$\bar{f}_1 = \bar{f}_1\eta_{GY}(G\varepsilon_Y) = \eta_X(GF\bar{f}_1)(G\varepsilon_Y) = \eta_X(GF\bar{f}_2)(G\varepsilon_Y) = \bar{f}_2\eta_{GY}(G\varepsilon_Y) = \bar{f}_2.$$

(3) ⇒ (1) For $X \in \mathrm{Ob}(\mathcal{C})$ and $Y \in \mathrm{Ob}(\mathcal{D})$ define the map

$$\varphi_{X,Y}\colon \quad \begin{array}{rcl} \mathcal{C}(X, GY) & \longrightarrow & \mathcal{D}(FX, Y) \\ g & \longmapsto & (Fg)\varepsilon_Y. \end{array}$$

By assumption, $\varphi_{X,Y}$ is a bijection. Suppose given morphisms $u\colon X' \to X$ in $\mathcal{C}$ and $v\colon Y \to Y'$ in $\mathcal{D}$. For $g \in \mathcal{C}(X, GY)$ we obtain using the naturality of ε

$$\begin{aligned} g\varphi_{X,Y}\mathcal{D}(Fu, v) &= ((Fg)\varepsilon_Y)\mathcal{D}(Fu, v) \\ &= (Fu)(Fg)\varepsilon_Y v \\ &= (Fu)(Fg)(FGv)\varepsilon_{Y'} \\ &= F(ug(Gv))\varepsilon_{Y'} \\ &= (ug(Gv))\varphi_{X',Y'} \\ &= g\mathcal{C}(u, Gv)\varphi_{X',Y'}. \end{aligned}$$

Hence the following diagram commutes.

$$\begin{array}{ccc} \mathcal{C}(X, GY) & \xrightarrow{\varphi_{X,Y}} & \mathcal{D}(FX, Y) \\ \big\downarrow{\scriptstyle \mathcal{C}(u,Gv)} & & \big\downarrow{\scriptstyle \mathcal{D}(Fu,v)} \\ \mathcal{C}(X', GY') & \xrightarrow{\varphi_{X',Y'}} & \mathcal{D}(FX', Y') \end{array}$$

Thus $\varphi := (\varphi_{X,Y}\colon \mathcal{C}(X, GY) \to \mathcal{D}(FX, Y))_{X \in \mathrm{Ob}(\mathcal{C}), Y \in \mathrm{Ob}(\mathcal{D})}$ constitutes a natural isomorphism $\varphi\colon \mathcal{C}(-, G(=)) \to \mathcal{D}(F(-), =)$, i.e. F is left adjoint to G. □

1.2 Graded modules and A_∞-algebras

Let R be a commutative ring.

All modules are left R-modules, all linear maps between modules are R-linear maps, all tensor products of modules are tensor products over R.

1.2.1 Graded modules

We first introduce *grading categories*, a formalism that allows us to handle classical A_∞-categories as A_∞-algebras over that grading category.

Definition 3 A *grading category* $\mathcal{Z} = (\mathcal{Z}, S, \lfloor - \rfloor)$ consists of a category $\mathcal{Z}$, a bijection $S\colon \operatorname{Mor}(\mathcal{Z}) \to \operatorname{Mor}(\mathcal{Z})$ between the morphisms of $\mathcal{Z}$, called *shift*, and a *degree function* $\lfloor - \rfloor\colon \operatorname{Mor}(\mathcal{Z}) \to \mathbf{Z}$, satisfying the following axioms.

(G1) For a morphism $z\colon x \to y$ from x to y in $\mathcal{Z}$ also its shift $zS\colon x \to y$ is a morphism from x to y.

(G2) For two composable morphisms $w\colon x \to x'$ and $z\colon x' \to x''$ in $\mathcal{Z}$ one has for the shift $(wz)S = (wS)z = w(zS)$ and for the degree $\lfloor wz \rfloor = \lfloor w \rfloor + \lfloor z \rfloor$.

(G3) For a morphism $z\colon x \to y$ in $\mathcal{Z}$ one has $\lfloor zS \rfloor = \lfloor z \rfloor + 1$.

For $k \in \mathbf{Z}$ we also write $z[k] := zS^k$.

In most examples, the grading category will be of the following form.

Example 4 Denote by $\mathbf{Z}$ the category with one object and morphisms given by the integers with addition as composition. Let $\mathcal{C}$ be a category.
Then the product category $\mathbf{Z} \times \mathcal{C}$ is a grading category with shift $(z, f)S = (z+1, f)S$ and degree function $\lfloor (z, f) \rfloor = z$ for $z \in \mathbf{Z}$ and $f \in \operatorname{Mor}(\mathcal{C})$.
In particular, we have the grading category $\mathbf{Z}$, which can be identified with $\mathbf{Z} \times \mathsf{1}$, where $\mathsf{1}$ is the trivial category with one object and one morphism.

Oftentimes, the category $\mathcal{C}$ will be a pair category over some set, which we define next.

Definition 5 Given a set X, the *pair category* over X is the category $\mathsf{Pair}(X)$ with objects $\operatorname{Ob}(\mathsf{Pair}(X)) = X$ and morphisms $\operatorname{Mor}(\mathsf{Pair}(X)) = X \times X$, where the only morphisms between $x \in X$ and $y \in X$ is the pair $(x, y) \in X \times X$.
The identity on $x \in X$ is the pair $(x, x)\colon x \to x$, for morphisms $(x, y)\colon x \to y$ and $(y, z)\colon y \to z$ their composite is the pair $(x, z)\colon x \to z$.

Definition 6 Let $\mathcal{Z}$ be a grading category. A $\mathcal{Z}$-*graded module* is a tuple $(M^z)_{z \in \operatorname{Mor}(\mathcal{Z})}$ of modules M^z. A *graded linear map* $f\colon M \to N$ is a tuple $(f^z)_{z \in \operatorname{Mor}(\mathcal{Z})}$ of linear maps $f^z\colon M^z \to N^z$.
Let M be a $\mathcal{Z}$-graded module and $z \in \operatorname{Mor}(\mathcal{Z})$. For $m \in M^z$ we call $\lfloor z \rfloor$ the *degree* of m. We often write $\lfloor m \rfloor := \lfloor z \rfloor$.
For graded linear maps $f\colon M \to N$ and $g\colon N \to P$, we define their composite $fg\colon M \to P$ by $(fg)^z := f^z g^z$. We obtain the category of grMod_0 of $\mathcal{Z}$-graded modules with graded linear maps.
The shift map S on the grading category $\mathcal{Z}$ induces the shift functor on the category grMod_0 of $\mathcal{Z}$-graded modules, which we will also denote by S.

$$\begin{array}{rrcl} S\colon & \mathsf{grMod}_0 & \longrightarrow & \mathsf{grMod}_0 \\ & M = (M^z)_{z \in \operatorname{Mor}(\mathcal{Z})} & \longmapsto & M^{[1]} = (M^{z[1]})_{z \in \operatorname{Mor}(\mathcal{Z})} \\ & (f = (f^z)_{z \in \operatorname{Mor}(\mathcal{Z})}\colon M \to N) & \longmapsto & (f^{[1]} = (f^{z[1]})_{z \in \operatorname{Mor}(\mathcal{Z})}\colon M^{[1]} \to N^{[1]}) \end{array}$$

Observe that the shift functor has a strict inverse, induced by the inverse shift S^{-1} on the grading category. For $k \in \mathbf{Z}$ we write $M^{[k]} := S^k(M)$ and $f^{[k]} := S^k(f)$.

A *graded linear map $f\colon M \to N$ of degree $p \in \mathbf{Z}$* is a graded linear map $f\colon M \to N^{[p]}$. Note that graded linear maps of degree 0 are just graded linear maps as defined above.

For graded linear maps $f\colon M \to N$ of degree p and $g\colon N \to P$ is a graded linear map of degree q we define their composite $fg\colon M \to P$ to be the graded linear map of degree $p+q$ given by the composite of $f\colon M \to N^{[p]}$ with $g^{[p]}\colon N^{[p]} \to P^{[p+q]}$. This defines the category grMod of $\mathcal{Z}$-graded modules with graded linear maps of arbitrary degree.

Let M and N be $\mathcal{Z}$-graded modules. The $\mathbf{Z}$-graded module $\mathsf{grHom}(M,N)$ of graded linear maps between M and N has at $p \in \mathbf{Z}$ the module $\mathsf{grHom}(M,N)^p$ of graded linear maps $f\colon M \to N$ of degree p.

To define a graded linear map $f\colon M \to N$ of degree p, we often write

$$\begin{array}{rccc} f\colon & M & \longrightarrow & N \\ f^z\colon & m & \longmapsto & mf^z \end{array}$$

to indicate that f is the graded linear map from M to N that is at $z \in \operatorname{Mor}(\mathcal{Z})$ given by the linear map $f^z\colon M^z \to N^{z[p]}$ that maps an element $m \in M^z$ to $mf^z \in N^{z[p]}$. We often write $mf := mf^z$.

Given $\mathcal{Z}$-graded modules and graded linear maps, we define submodules, factor modules, kernels, cokernels and images degreewise. This way, the category dgMod of $\mathcal{Z}$-graded modules is an abelian category.

Similarly, we say that a graded linear map $f\colon M \to N$ is injective, surjective resp. bijective, if f^z is injective, surjective resp. bijective for each $z \in \operatorname{Mor}(\mathcal{Z})$.

Definition 7 Using the composition of morphisms on $\mathcal{Z}$, we can define the tensor product of $\mathcal{Z}$-graded modules. Suppose given $\mathcal{Z}$-graded modules $M_1, \ldots, M_k$. Their tensor product is defined as the $\mathcal{Z}$-graded module given at $z \in \operatorname{Mor}(\mathcal{Z})$ by

$$(M_1 \otimes \ldots \otimes M_k)^z = \bigoplus_{z = w_1 \cdots w_k} M_1^{w_1} \otimes \ldots \otimes M_k^{w_k}.$$

Here, the direct sum runs over all factorisations of z into k factors $w_1, \ldots, w_k$ in the grading category $\mathcal{Z}$.

For the tensor product of graded linear maps, we impose the *Koszul sign rule.* Suppose given graded linear maps $f_i\colon M_i \to N_i$ of degree p_i for $1 \le i \le k$. Then we define their tensor product

$$f_1 \otimes \ldots \otimes f_k\colon M_1 \otimes \ldots \otimes M_k \to N_1 \otimes \ldots \otimes N_k$$

as the graded linear map of degree $p_1 + \ldots + p_k$ defined at $z \in \operatorname{Mor}(\mathcal{Z})$ by

$$(m_1 \otimes \ldots \otimes m_k)(f_1 \otimes \ldots \otimes f_k)^z := (-1)^{\sum_{1 \le i < j \le k} p_i \lfloor w_j \rfloor}(m_1 f_1^{w_1} \otimes \ldots \otimes m_k f_k^{w_k}),$$

where $m_i \in M_i^{w_i}$ and $z = w_1 \cdots w_k$ is a factorisation of z into k factors w_i in $\mathcal{Z}$. We remark that the Koszul sign also appears when one composes tensor products of graded linear maps. Suppose we also have graded linear maps $g_i\colon N_i \to P_i$ of degree q_i for $1 \le i \le k$. Then the following formula holds

$$(f_1 \otimes \ldots \otimes f_k)(g_1 \otimes \ldots \otimes g_k) = (-1)^{\sum_{1 \le i < j \le k} q_i p_j}(f_1 g_1 \otimes \ldots \otimes f_k g_k).$$

Remark 8 Let $\dot{R}$ be the $\mathcal{Z}$-graded module with

$$\dot{R}^z := \begin{cases} R & \text{if } z = \mathrm{id}_X \text{ for } X \in \mathrm{Ob}(\mathcal{Z}) \\ 0 & \text{if } z \text{ is not an identity.} \end{cases}$$

Given a $\mathcal{Z}$-graded module M and $z \in \mathrm{Mor}(\mathcal{Z})$, where $z\colon X \to Y$ with $X, Y \in \mathrm{Ob}(\mathcal{Z})$, we have

$$(\dot{R} \otimes M)^z = \bigoplus_{z = w_1 w_2} \dot{R}^{w_1} \otimes M^{w_2} = \dot{R}^{\mathrm{id}_X} \otimes M^z = R \otimes M^z$$

and similarly

$$(M \otimes \dot{R})^z = \bigoplus_{z = w_1 w_2} M^{w_1} \otimes \dot{R}^{w_2} = M^z \otimes \dot{R}^{\mathrm{id}_Y} = M^z \otimes R$$

Hence the isomorphisms of modules $R \otimes M^z \xrightarrow{\sim} M^z$ and $M^z \otimes R \xrightarrow{\sim} M^z$ define the following canonical isomorphisms of $\mathcal{Z}$-graded modules, the *tensor unit isomorphisms*

$$\begin{array}{rccc} \lambda\colon & \dot{R} \otimes M & \xrightarrow{\sim} & M \\ \lambda^z\colon & (r \otimes m) & \longmapsto & rm \end{array} \quad \text{and} \quad \begin{array}{rccc} \rho\colon & M \otimes \dot{R} & \xrightarrow{\sim} & M \\ \rho^z\colon & (m \otimes r) & \longmapsto & rm \end{array}$$

We will identify along both isomorphisms λ and ρ.
For a $\mathcal{Z}$-graded module M we write $M^{\otimes 0} := \dot{R}$, and for a graded linear map $f\colon M \to N$ of degree 0 we let $f^{\otimes 0} := \mathrm{id}_{\dot{R}}\colon \dot{R} \to \dot{R}$.

1.2.2 Differential graded modules and cohomology

We endow $\mathcal{Z}$-graded modules with differentials and obtain differential graded modules. In the case of $\mathbf{Z}$-graded modules, this gives the usual definition of a complex.

Definition 9 Let $\mathcal{Z}$ be a grading category. A *differential $\mathcal{Z}$-graded module* $M = (M, d)$ is a $\mathcal{Z}$-graded module M together with a graded linear map $d\colon M \to M$ of degree 1, called *differential*, that satisfies $dd = 0$.
A *morphism of differential $\mathcal{Z}$-graded modules* is a graded linear map $f\colon M \to N$ of degree 0 that satisfies $fd_N = d_M f$. Composition is given by the composition in grMod. This defines the category dgMod of differental $\mathcal{Z}$-graded modules and morphisms of differential graded modules between them.
The category of differential $\mathcal{Z}$-graded modules is an abelian category.

For differential graded modules, we can define cohomology.

Definition 10 Let $M = (M, d)$ be a differential $\mathcal{Z}$-graded module.

(1) The *cohomology module* of M is the $\mathcal{Z}$-graded module $\mathrm{H}M$ that is at $z \in \mathrm{Mor}(\mathcal{Z})$ given by the factor module

$$(\mathrm{H}M)^z := \ker(d^z)/\operatorname{im}(d^{z[-1]}).$$

This is well-defined, since $dd = 0$ implies that $d^{z[-1]} d^z = 0$, i.e. $\operatorname{im}(d^{z[-1]}) \subseteq \ker(d^z)$ for $z \in \mathrm{Mor}(\mathcal{Z})$.

(2) Suppose given differential $\mathcal{Z}$-graded modules $M = (M, d_M)$ and $N = (N, d_N)$ and a morphism of differential $\mathcal{Z}$-graded modules $f\colon M \to N$ between them.

We define a $\mathcal{Z}$-graded linear map $\mathrm{H}f\colon \mathrm{H}M \to \mathrm{H}N$ of degree 0 by

$$\begin{array}{rrcl} \mathrm{H}f\colon & \mathrm{H}M & \longrightarrow & \mathrm{H}N \\ (\mathrm{H}f)^z\colon & m + \mathrm{im}(d_M^{z[-1]}) & \longmapsto & mf^z + \mathrm{im}(d_N^{z[-1]}). \end{array}$$

This is well-defined, since for $m \in \mathrm{im}(d_M^{z[-1]})$, i.e. $m = nd^{z[-1]}$ for some $n \in M^{z[-1]}$ we have

$$mf^z = nd_M^{z[-1]} f^z = nf^{z-1} d_N^{z[-1]} \in \mathrm{im}(d_N^{z[-1]}).$$

The morphism f is a *quasiisomorphism* if $\mathrm{H}f$ is an isomorphism.

Remark 11 Cohomology of $\mathcal{Z}$-graded modules defines a functor

$$\begin{array}{rrcl} \mathrm{H}\colon & \mathsf{dgMod} & \longrightarrow & \mathsf{grMod} \\ & M & \longmapsto & \mathrm{H}M \\ & (f\colon M \to N) & \longmapsto & (\mathrm{H}f\colon \mathrm{H}M \to \mathrm{H}N), \end{array}$$

cf. Definition 10.

Proof. Suppose given a differential $\mathcal{Z}$-graded module $M = (M, d_M)$. For $z \in \mathrm{Mor}(\mathcal{Z})$ and $m \in \ker(d_M^z)$ we have

$$(m + \mathrm{im}(d_M^{z[-1]}))\mathrm{H}\,\mathrm{id}_M = m + \mathrm{im}(d_M^{z[-1]}) = (m + \mathrm{im}(d_M^{z[-1]}))\,\mathrm{id}_{\mathrm{H}M}.$$

Hence $\mathrm{H}\,\mathrm{id}_M = \mathrm{id}_{\mathrm{H}M}$. Suppose given morphisms of differential $\mathcal{Z}$-graded modules $f\colon M \to N$ and $g\colon N \to P$. For $z \in \mathrm{Mor}(\mathcal{Z})$ and $m \in \ker(d_M^z)$ we have

$$\begin{aligned} (m + \mathrm{im}(d_M^{z[-1]}))(\mathrm{H}f)(\mathrm{H}g) = (mf + \mathrm{im}(d_N^{z[-1]}))\mathrm{H}g &= mfg + \mathrm{im}(d_P^{z[-1]}) \\ &= (m + \mathrm{im}(d_M^{z[-1]}))\mathrm{H}(fg). \end{aligned}$$

Hence $\mathrm{H}(fg) = (\mathrm{H}f)(\mathrm{H}g)$. We conclude that H is a functor. □

Lemma 12 *Suppose given a differential $\mathcal{Z}$-graded module (M, d). We endow the tensor product $M^{\otimes k}$ as $\mathcal{Z}$-graded modules with the differential*

$$\delta = \sum_{r=1}^{k} \mathrm{id}^{\otimes(r-1)} \otimes d \otimes \mathrm{id}^{\otimes(k-r)}.$$

This turns $(M^{\otimes k}, \delta)$ into a differential $\mathcal{Z}$-graded module.

Proof. We show that δ is indeed a differential on $M^{\otimes k}$. Note that since the differential d on

M is of degree 1, we have to make use of the Koszul sign rule.

$$\begin{aligned}
\delta\delta &= \left(\sum_{r=1}^{k} \mathrm{id}^{\otimes(r-1)} \otimes d \otimes \mathrm{id}^{\otimes(k-r)}\right)\left(\sum_{s=1}^{k} \mathrm{id}^{\otimes(s-1)} \otimes d \otimes \mathrm{id}^{\otimes(k-s)}\right) \\
&= \sum_{1\le r<s\le k} \big(\mathrm{id}^{\otimes(r-1)} \otimes d \otimes \mathrm{id}^{\otimes(k-r)}\big)\big(\mathrm{id}^{\otimes(s-1)} \otimes d \otimes \mathrm{id}^{\otimes(k-s)}\big) \\
&\quad + \sum_{1\le t\le k} \big(\mathrm{id}^{\otimes(t-1)} \otimes d \otimes \mathrm{id}^{\otimes(k-t)}\big)\big(\mathrm{id}^{\otimes(t-1)} \otimes d \otimes \mathrm{id}^{\otimes(k-t)}\big) \\
&\quad + \sum_{1\le s<r\le k} \big(\mathrm{id}^{\otimes(r-1)} \otimes d \otimes \mathrm{id}^{\otimes(k-r)}\big)\big(\mathrm{id}^{\otimes(s-1)} \otimes d \otimes \mathrm{id}^{\otimes(k-s)}\big) \\
&= \sum_{1\le r<s\le k} \big(\mathrm{id}^{\otimes(r-1)} \otimes d \otimes \mathrm{id}^{\otimes(s-r-1)} \otimes d \otimes \mathrm{id}^{\otimes(k-s)}\big) \\
&\quad + \sum_{1\le t\le k} \big(\mathrm{id}^{\otimes(t-1)} \otimes dd \otimes \mathrm{id}^{\otimes(k-t)}\big) \\
&\quad - \sum_{1\le s<r\le k} \big(\mathrm{id}^{\otimes(s-1)} \otimes d \otimes \mathrm{id}^{\otimes(r-s-1)} \otimes d \otimes \mathrm{id}^{\otimes(k-r)}\big) \\
&= 0.
\end{aligned}$$

□

1.2.3 A_∞-algebras

Definition 13 An $\mathrm{A}_\infty^{[1]}$-algebra $(A, (\mu_k)_{k\ge1})$ over $\mathcal{Z}$ is a $\mathcal{Z}$-graded module A together with a tuple of $\mathcal{Z}$-graded linear maps $\mu_k\colon (A^{[1]})^{\otimes k} \to A^{[1]}$ of degree 1 that satisfy the *Stasheff equations* for $k \ge 1$.

$$0 = \sum_{\substack{r+s+t=k \\ r,t\ge0,\, s\ge1}} (\mathrm{id}^{\otimes r} \otimes \mu_s \otimes \mathrm{id}^{\otimes t})\mu_{r+1+t}$$

A morphism of $\mathrm{A}_\infty^{[1]}$-algebras $\varphi\colon A \to B$ is a tuple $\varphi = (\varphi_k)_{k\ge1}$ of $\mathcal{Z}$-graded linear maps $\varphi_k\colon (A^{[1]})^{\otimes k} \to B^{[1]}$ of degree 1 that satisfy the following *Stasheff equations for morphisms* for $k \ge 1$.

$$\sum_{\substack{r+s+t=k \\ r,t\ge0,\, s\ge1}} (\mathrm{id}^{\otimes r} \otimes \mu_s \otimes \mathrm{id}^{\otimes t})\varphi_{r+1+t} = \sum_{1\le r\le k} \sum_{\substack{i_1+\ldots+i_r=k \\ i_1,\ldots,i_r\ge1}} (\varphi_{i_1} \otimes \ldots \otimes \varphi_{i_r})\mu_r$$

For $k = 1$ the Stasheff equation becomes $\mu_1\mu_1 = 0$. It follows that $(A^{[1]}, \mu_1)$ is a differential $\mathcal{Z}$-graded module. The *cohomology module* of the $\mathrm{A}_\infty^{[1]}$-algebra $(A, (\mu_k)_{k\ge1})$ is the cohomology module of the differential $\mathcal{Z}$-graded module $(A^{[1]}, \mu_1)$, cf. Definition 10.(1).

For $k = 1$ the Stasheff equation for morphisms becomes $\mu_1\varphi_1 = \varphi_1\mu_1$, i.e. for a morphism of $\mathrm{A}_\infty^{[1]}$-algebras $\varphi\colon A \to B$ the first component $\varphi_1\colon A^{[1]} \to B^{[1]}$ is a morphism of differential $\mathcal{Z}$-graded modules between $(A^{[1]}, \mu_1)$ and $(B^{[1]}, \mu_1)$.

An $\mathrm{A}_\infty^{[1]}$-*quasiisomorphism* is a morphism of $\mathrm{A}_\infty^{[1]}$-algebras $\varphi\colon A \to B$ such that $\varphi_1\colon A^{[1]} \to B^{[1]}$ is a quasiisomorphism of differential $\mathcal{Z}$-graded modules, cf. Definition 10.(2).

Definition 14 (cf. [Sta63]) An A_∞-algebra $(A, (\mathsf{m}_k)_{k\ge1})$ is a $\mathcal{Z}$-graded module A together with a tuple of graded linear maps $\mathsf{m}_k\colon A^{\otimes k} \to A$ of degree $2-k$ satisfying the Stasheff equations for $k \geq 1$

$$0 = \sum_{\substack{r+s+t=k \\ r,t\geq 0,\, s\geq 1}} (-1)^{r+st} (\mathrm{id}^{\otimes r} \otimes \mathsf{m}_s \otimes \mathrm{id}^{\otimes t}) \mathsf{m}_{r+1+t}.$$

A morphism $f\colon A \to B$ of A_∞-algebras is a tuple $(f_k)_{k\geq 1}$ of graded linear maps $f_k\colon A^{\otimes k} \to B$ of degree $1-k$ satisfying

$$\sum_{\substack{r+s+t=k \\ r,t\geq 0,\, s\geq 1}} (-1)^{r+st} (\mathrm{id}^{\otimes r} \otimes \mathsf{m}_s \otimes \mathrm{id}^{\otimes t}) f_{r+1+t}$$
$$= \sum_{1\leq r\leq k} \sum_{\substack{i_1+\ldots+i_r=k \\ i_1,\ldots,i_r\geq 1}} (-1)^{\sum_{1\leq p<q\leq r}(1-i_p)i_q} (f_{i_1} \otimes \ldots \otimes f_{i_r}) \mathsf{m}_r.$$

Remark 15 (1) Let $(A, (\mathsf{m})_{k\geq 1})$ be an A_∞-algebra. Consider the graded linear map $\omega\colon A \to A^{[1]}$ of degree -1 given by $\omega^z := \mathrm{id}\colon A^z \to (A^{[1]})^{z[-1]} = A^z$ at $z \in \mathrm{Mor}(\mathcal{Z})$. One can conjugate the maps m_k of degree $2-k$ to graded linear maps

$$\mu_k := (\omega^{-1})^{\otimes k} \mathsf{m}_k \omega\colon (A^{[1]})^{\otimes k} \to A^{[1]}$$

of degree 1. By the Koszul sign rule, the μ_k satisfy the Stasheff equation from Definition 13, i.e. $(A^{[1]}, (\mu_k)_{k\geq 1})$ is an $\mathrm{A}_\infty^{[1]}$-algebra over $\mathcal{Z}$. This way, an A_∞-algebra $(A, (\mathsf{m}_k)_{k\geq 1})$ corresponds to an $\mathrm{A}_\infty^{[1]}$-algebra $(A^{[1]}, (\mu_k)_{k\geq 1})$.

Similarly, conjugating the graded linear maps $f_k\colon A^{\otimes k} \to B$ of degree $1-k$ with ω yields graded linear maps $\varphi_k\colon (A^{[1]})^{\otimes k} \to B^{[1]}$ of degree 0, which then satisfy the Stasheff equation for morphisms of $\mathrm{A}_\infty^{[1]}$-algebras from the definition above. That is, there is a bijection between A_∞-algebra morphisms from $(A, (\mathsf{m}_k)_{k\geq 1})$ to $(B, (\mathsf{m}_k)_{k\geq 1})$ and $\mathrm{A}_\infty^{[1]}$-algebra morphisms between $(A^{[1]}, (\mu_k)_{k\geq 1})$ and $(B^{[1]}, (\mu_k)_{k\geq 1})$.

As in the case of $\mathrm{A}_\infty^{[1]}$-algebras, an A_∞-algebra $(A, (\mathsf{m}_k)_{k\geq 1})$ gives rise to a differential $\mathcal{Z}$-graded module (A, m_1). An A_∞-morphism $f\colon A \to B$ is called an A_∞*-quasiisomorphism* if $f_1\colon A \to B$ is a quasiisomorphism of differential $\mathcal{Z}$-graded modules.

Since ω is an isomorphism, $f\colon A \to B$ is an A_∞-quasiisomorphism if and only if the corresponding $\mathrm{A}_\infty^{[1]}$-algebra morphism $\varphi\colon A^{[1]} \to B^{[1]}$ is an $\mathrm{A}_\infty^{[1]}$-quasiisomorphism.

(2) The case of classical A_∞-algebras is included in our definition using the grading category $\mathbf{Z}$. The case of A_n-categories in the sense of [Kel01] or [Sei08] is included using a grading category of the form $\mathbf{Z} \times \mathsf{Pair}(X)$, where X is the set of objects of the A_∞-category.

1.3 Coalgebras

Let R be a commutative ring.

All modules are left R-modules, all linear maps between modules are R-linear maps, all tensor products of modules are tensor products over R.

Fix a grading category $\mathcal{Z}$. Unless stated otherwise, by *graded* we mean $\mathcal{Z}$-*graded*.

In this section, our first aim is to review the classical *Bar construction*, cf. §1.3.3 below. We will obtain a full and faithful functor

$$\mathrm{Bar}\colon \quad \mathrm{A}_\infty\text{-}\mathsf{alg} \;\; \to \;\; \mathsf{dgCoalg}\,.$$

The image of Bar is the category $\mathsf{dtCoalg}$ of differential graded tensor coalgebras.
The coalgebras in $\mathsf{dtCoalg}$ will not be equipped with a counit. However, we describe how one can construct a counital coalgebra out of an arbitrary coalgebra in a functorial way and then apply the general construction to tensor coalgebras, cf. §1.3.4 and §1.3.5 below. This simplifies formulas and avoids case distinctions, cf. e.g. Lemma 37.

1.3.1 Definitions

Definition 16

(1) A *graded coalgebra* $C = (C, \Delta)$ is a graded module C with a graded linear map $\Delta\colon C \to C\otimes C$ of degree 0, the *comultiplication*, that is coassociative, i.e. $\Delta(\mathrm{id}\otimes\Delta) = \Delta(\Delta\otimes\mathrm{id})$.

$$\begin{array}{ccc} C & \xrightarrow{\;\;\Delta\;\;} & C\otimes C \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta\otimes\mathrm{id}} \\ C\otimes C & \xrightarrow{\mathrm{id}\otimes\Delta} & C\otimes C\otimes C \end{array}$$

(2) Let $C = (C, \Delta_C)$ and $D = (D, \Delta_D)$ be graded coalgebras. A *morphism of graded coalgebras* is a graded linear map $f\colon C \to D$ of degree 0 that satisfies $f\Delta_D = \Delta_C(f\otimes f)$.

$$\begin{array}{ccc} C & \xrightarrow{\;\;f\;\;} & D \\ \downarrow{\scriptstyle \Delta_C} & & \downarrow{\scriptstyle \Delta_D} \\ C\otimes C & \xrightarrow{f\otimes f} & D\otimes D \end{array}$$

With composition and identity as in the category of graded modules we obtain the category $\mathsf{grCoalg}$ of graded coalgebras and morphisms of graded coalgebras between them.

(3) A *counital graded coalgebra* $C = (C, \Delta, \varepsilon)$ is a graded coalgebra (C, Δ) with a graded linear map $\varepsilon\colon C \to \dot{R}$ of degree 0, the *counit*, such that $\Delta(\mathrm{id}\otimes\varepsilon) = \mathrm{id}_C = \Delta(\varepsilon\otimes\mathrm{id})$.

$$\begin{array}{ccccc} C\otimes\dot{R} & \xleftarrow{\mathrm{id}\otimes\varepsilon} & C\otimes C & \xrightarrow{\varepsilon\otimes\mathrm{id}} & \dot{R}\otimes C \\ & \Vert & \uparrow{\scriptstyle \Delta} & \Vert & \\ & & C & & \end{array}$$

(4) Let $C = (C, \Delta_C, \varepsilon_C)$ and $D = (D, \Delta_D, \varepsilon_D)$ be counital graded coalgebras. A *morphism of counital graded coalgebras* is a morphism of graded coalgebras $f\colon C \to D$ such that $f\varepsilon_D = \varepsilon_C$.

$$\begin{array}{ccc} C & \xrightarrow{\;f\;} & D \\ & {\scriptstyle \varepsilon_C}\searrow & \downarrow{\scriptstyle \varepsilon_D} \\ & & \dot{R} \end{array}$$

With composition and identity as in the category of graded modules we obtain the category $\mathsf{grCoalg}_*$ of counital graded coalgebras and morphisms of counital graded coalgebras between them.

(5) A *differential graded coalgebra* $C = (C, \Delta, m)$ is a graded coalgebra (C, Δ) with a differential $m \colon C \to C$, i.e. m is a graded linear map of degree 1 with $mm = 0$, such that $m\Delta = \Delta(\mathrm{id} \otimes m + m \otimes \mathrm{id})$.

Note that (C, m) is a differential graded module and $\Delta \colon C \to C \otimes C$ is a morphism of differential graded modules, cf. Lemma 12.

(6) Let $C = (C, \Delta_C, m_C)$ and $D = (D, \Delta_D, m_C)$ be differential graded coalgebras. A *morphism of differential graded coalgebras* from C to D is a graded linear map $f \colon C \to D$ of degree 0 that is both a morphism of differential graded modules and a morphism of graded coalgebras. That is, it satisfies both $fm_C = m_C f$ and $f\Delta_D = \Delta_C(f \otimes f)$.

With composition and identity as in the category of graded modules we obtain the category $\mathsf{dgCoalg}$ of differential graded coalgebras and morphisms of differential graded coalgebras between them.

We will often drop the index for comultiplication and differential, i.e. we will just write Δ for the comultiplication of a graded coalgebra and m for the differential on a differential graded coalgebra.

Remark 17 Let $C = (C, \Delta, m)$ and $D = (D, \Delta, m)$ be differential graded coalgebras. Let $f \colon C \to D$ be a morphism of differential graded coalgebras.

Then f is an isomorphism of differential graded coalgebras if and only if it is an isomorphism of graded coalgebras.

Proof. Suppose that f is an isomorphism of graded coalgebras. Let $f^{-1} \colon D \to C$ be the inverse. Then f^{-1} is a morphism of graded coalgebras. Moreover, using that f is a morphism of differential graded coalgebras we obtain

$$f^{-1}m = f^{-1}mff^{-1} = f^{-1}fmf^{-1} = mf^{-1}.$$

Hence f^{-1} is a morphism of differential graded coalgebras, thus f is an isomorphism of differential graded coalgebras.

The other direction is clear. $\square$

1.3.2 Tensor coalgebras

Definition 18 Let A be a graded module.

Define the graded module $TA = \bigoplus_{k \geq 1} A^{\otimes k}$. Let $\iota_k \colon A^{\otimes k} \to TA$ be the inclusion into the k-th summand and let $\pi_k \colon TA \to A^{\otimes k}$ the projection onto the k-th summand.

Define the graded linear map $\Delta \colon TA \to TA \otimes TA$ on the summand $k \geq 1$ by

$$\begin{array}{rccc} \iota_k\Delta \colon & A^{\otimes k} & \longrightarrow & TA \otimes TA \\ (\iota_k\Delta)^z \colon & a_1 \otimes \ldots \otimes a_k & \longmapsto & \displaystyle\sum_{\substack{i+j=k \\ i,j \geq 1}} (a_1 \otimes \ldots \otimes a_k)(\iota_i \otimes \iota_j)^z. \end{array}$$

Then (TA, Δ) is a graded coalgebra, the *tensor coalgebra* over A.

From the definition of the comultiplication and the universal property of the kernel, we can conclude the following remark.

Remark 19 The kernel of Δ is the first summand $A^{\otimes 1}$. In particular, we have $\iota_1 \Delta = 0$. Moreover, a graded linear map $f \colon TA \to TB$ with $f\Delta = 0$ has its image in the first summand, i.e. $f\Delta = 0$ if and only if $f = f\pi_1\iota_1$.

Remark 20 Let TA be the tensor coalgebra over a graded module A. For $k, \ell_1, \ell_2 \geq 1$ the comultiplication Δ on TA satisfies the following.

(1) $\iota_k \Delta(\pi_{\ell_1} \otimes \pi_{\ell_2}) = \left\{\begin{matrix} \mathrm{id}^{\otimes k} & \text{for } k = \ell_1 + \ell_2 \\ 0 & \text{else} \end{matrix}\right\} : A^{\otimes k} \to A^{\otimes \ell_1} \otimes A^{\otimes \ell_2} = A^{\otimes(\ell_1+\ell_2)}$.

(2) $\Delta(\pi_{\ell_1} \otimes \pi_{\ell_2}) = \pi_{\ell_1+\ell_2}$

(3) $\iota_k \Delta = \sum_{\substack{i+j=k \\ i,j \geq 1}} \iota_i \otimes \iota_j$

Proof. (1) Let $z \in \mathrm{Mor}(\mathcal{Z})$ and let $a_1 \otimes \ldots \otimes a_k \in \left(A^{\otimes k}\right)^z$. Then

$$(a_1 \otimes \ldots \otimes a_k)\iota_k^z \Delta^z(\pi_{\ell_1} \otimes \pi_{\ell_2})^z = \sum_{\substack{i+j=k \\ i,j \geq 1}} (a_1 \otimes \ldots \otimes a_k)(\iota_i \otimes \iota_j)^z(\pi_{\ell_1} \otimes \pi_{\ell_2})^z$$
$$= \sum_{\substack{i+j=k \\ i,j \geq 1}} (a_1 \otimes \ldots \otimes a_k)(\iota_i\pi_{\ell_1} \otimes \iota_j\pi_{\ell_2})^z.$$

If $\ell_1 + \ell_2 = k$, then only the summand with $i = \ell_1$ and $j = \ell_2$ above is non-zero and equals $a_1 \otimes \ldots \otimes a_k$, it follows that $\iota_k\Delta(\pi_{\ell_1} \otimes \pi_{\ell_2}) = \mathrm{id}^{\otimes k}$.
If $\ell_1 + \ell_2 \neq k$, then $i = \ell_1$ and $j = \ell_2$ can not hold both, i.e. the sum above is zero and it follows that $\iota_k\Delta(\pi_{\ell_1} \otimes \pi_{\ell_2}) = 0$.
(2) For $k \geq 1$ we have using (1) that

$$\iota_k\Delta(\pi_{\ell_1} \otimes \pi_{\ell_2}) = \left\{\begin{matrix} \mathrm{id}^{\otimes k} & \text{for } k = \ell_1 + \ell_2 \\ 0 & \text{else} \end{matrix}\right\} = \iota_k \pi_{\ell_1+\ell_2}.$$

(3) This is the definition of the comultiplication Δ. □

Notation 21 Given a graded linear map $f \colon TA \to TB$ between two tensor coalgebras over graded modules A and B, we write $f_{k,\ell} := \iota_k f \pi_\ell \colon A^{\otimes k} \to B^{\otimes \ell}$ for $k, \ell \geq 1$.
Similarly, for a graded linear map $\varphi \colon TA \to B$ and $k \geq 1$ we write $\varphi_k := \iota_k\varphi \colon A^{\otimes k} \to B$.
Conversely, given graded linear maps $f_{k,\ell} \colon A^{\otimes k} \to B^{\otimes \ell}$ for $k, \ell \geq 1$ such that for all $k \geq 1$ the set $\{\ell \in \mathbf{N} : f_{k,\ell} \neq 0\}$ is finite, there is a unique graded linear map $f \colon TA \to TB$ with $f_{k,\ell} = \iota_k f \pi_\ell$. Note that the finiteness assumption is required since the tensor coalgebra is defined as an infinite direct sum (i.e. an infinite coproduct).

In particular, given two graded linear maps $f\colon TA \to TB$ and $g\colon TB \to TC$ between tensor coalgebras over graded modules A, B and C the (k,ℓ)-entry for the composite is given by

$$(fg)_{k,\ell} = \sum_{j\geq 1} f_{k,j} g_{j,\ell}.$$

Note that the above conditions on f and g ensure that the sum is finite. Oftentimes, we consider such graded linear maps with $f_{k,\ell} = g_{k,\ell} = 0$ for $k < \ell$. In this case, the formula above becomes

$$(fg)_{k,\ell} = \sum_{j=\ell}^{k} f_{k,j} g_{j,\ell}.$$

We will make use of this matrix calculus without further comment.

Lemma 22 *Let A and B be graded modules. Then the following hold.*

(1) *Consider the map*

$$\begin{array}{rrcl} \beta := \beta_{\mathrm{Coalg}}\colon & \mathsf{grCoalg}(TA,TB) & \longrightarrow & \mathsf{grHom}(TA,B)^0 \\ & f & \longmapsto & f\pi_1 \end{array}$$

from the set $\mathsf{grCoalg}(TA,TB)$ of morphisms of graded coalgebras $TA \to TB$ to the set $\mathsf{grHom}(TA,B)^0$ of graded linear maps $TA \to B$ of degree 0.

Consider the map $\alpha := \alpha_{\mathrm{Coalg}}\colon \mathsf{grHom}(TA,B)^0 \to \mathsf{grCoalg}(TA,TB)$ that is for a graded linear map $\varphi \in \mathsf{grHom}(TA,B)^0$ for $k,\ell \geq 1$ given by

$$(\varphi\alpha)_{k,\ell} := \sum_{\substack{i_1+\ldots+i_\ell=k \\ i_1,\ldots,i_\ell\geq 1}} \varphi_{i_1} \otimes \ldots \otimes \varphi_{i_\ell}.$$

Then α and β are mutually inverse bijections.

In particular, for a coalgebra morphism $f\colon TA \to TB$ between tensor coalgebras the following formula holds for $k,\ell \geq 1$.

$$f_{k,\ell} = \sum_{\substack{i_1+\ldots+i_\ell=k \\ i_1,\ldots,i_\ell\geq 1}} f_{i_1,1} \otimes \ldots \otimes f_{i_\ell,1}$$

Note that this implies that $f_{k,k} = f_{1,1}^{\otimes k}$.

(2) *Let $\mathrm{Coder}(TA,TA)^{1,(\mathrm{id},\mathrm{id})}$ be the module of* coderivations *on TA, i.e. the module of graded linear maps $m\colon TA \to TA$ of degree 1 that satisfy $m\Delta = \Delta(\mathrm{id}\otimes m + m \otimes \mathrm{id})$. Consider the linear map*

$$\begin{array}{rrcl} \beta := \beta_{\mathrm{Coder}}\colon & \mathrm{Coder}(TA,TA)^{1,(\mathrm{id},\mathrm{id})} & \longrightarrow & \mathsf{grHom}(TA,A)^1 \\ & m & \longmapsto & m\pi_1 \end{array}$$

from the module of coderivations on TA to the module $\mathsf{grHom}(TA,A)^1$ of graded linear maps $TA \to A$ of degree 1.

Consider the linear map $\alpha := \alpha_{\mathrm{Coder}} \colon \mathsf{grHom}(TA, A)^1 \to \mathrm{Coder}(TA, TA)^{1,(\mathrm{id},\mathrm{id})}$ *that is for a graded linear map* $\mu \in \mathsf{grHom}(TA, A)^1$ *for* $k, \ell \geq 1$ *given by*

$$(\mu\alpha)_{k,\ell} := \sum_{\substack{r+s+t=k\\ r+1+t=\ell\\ (r,s,t)\geq(0,1,0)}} \mathrm{id}^{\otimes r} \otimes \mu_s \otimes \mathrm{id}^{\otimes t}\,.$$

Then α *and* β *are mutually inverse linear isomorphisms.*

In particular, for a coderivation $m \colon TA \to TA$ *on a tensor coalgebra the following formula holds for* $k, \ell \geq 1$.

$$m_{k,\ell} = \sum_{\substack{r+s+t=k\\ r+1+t=\ell\\ (r,s,t)\geq(0,1,0)}} \mathrm{id}^{\otimes r} \otimes m_{s,1} \otimes \mathrm{id}^{\otimes t}$$

Note that this implies that $m_{k,k} = \sum_{i=0}^{k-1} \mathrm{id}^{\otimes i} \otimes m_{1,1} \otimes \mathrm{id}^{\otimes(k-i-1)}$.

Concerning the notation $\mathrm{Coder}(TA, TA)^{1,(\mathrm{id},\mathrm{id})}$ for the module of coderivations on TA, cf. also Definition 34 below. Moreover, in Lemma 37 below we prove a generalisation of the above Lemma 22.(2) to general (f, g)-coderivations.

Proof. (1) *We show that* α *is well-defined.* That is, given a graded linear map $\varphi \colon TA \to B$ of degree 0 we show that $\varphi\alpha$ is a coalgebra morphism.

It suffices to show that $\iota_k(\varphi\alpha)\Delta(\pi_{\ell_1} \otimes \pi_{\ell_2}) = \iota_k\Delta\big((\varphi\alpha) \otimes (\varphi\alpha)\big)(\pi_{\ell_1} \otimes \pi_{\ell_2})$ for all $k, \ell_1, \ell_2 \geq 1$. Using Remark 20, the left-hand side gives

$$\begin{aligned}\iota_k(\varphi\alpha)\Delta(\pi_{\ell_1} \otimes \pi_{\ell_2}) &= \iota_k(\varphi\alpha)\pi_{\ell_1+\ell_2}\\ &= (\varphi\alpha)_{k,\ell_1+\ell_2}\end{aligned}$$

while the right-hand side gives

$$\begin{aligned}\iota_k\Delta\big((\varphi\alpha) \otimes (\varphi\alpha)\big)(\pi_{\ell_1} \otimes \pi_{\ell_2}) &= \sum_{\substack{i+j=k\\ i,j\geq 1}} (\iota_i \otimes \iota_j)\big((\varphi\alpha) \otimes (\varphi\alpha)\big)(\pi_{\ell_1} \otimes \pi_{\ell_2})\\ &= \sum_{\substack{i+j=k\\ i,j\geq 1}} (\varphi\alpha)_{i,\ell_1} \otimes (\varphi\alpha)_{j,\ell_2}.\end{aligned}$$

We obtain

$$\begin{aligned}(\varphi\alpha)_{k,\ell_1+\ell_2} &= \sum_{\substack{i_1+\ldots+i_{\ell_1}+j_1+\ldots+j_{\ell_2}=k\\ i_1,\ldots,i_{\ell_1},j_1,\ldots,j_{\ell_2}\geq 1}} \varphi_{i_1} \otimes \ldots \otimes \varphi_{i_{\ell_1}} \otimes \varphi_{j_1} \otimes \ldots \otimes \varphi_{j_{\ell_2}}\\ &= \sum_{\substack{i+j=k\\ i,j\geq 1}} \sum_{\substack{i_1+\ldots+i_{\ell_1}=i\\ i_1,\ldots,i_{\ell_1}\geq 1}} \sum_{\substack{j_1+\ldots+j_{\ell_2}=j\\ j_1,\ldots,j_{\ell_2}\geq 1}} \varphi_{i_1} \otimes \ldots \otimes \varphi_{i_{\ell_1}} \otimes \varphi_{j_1} \otimes \ldots \otimes \varphi_{j_{\ell_2}}\\ &= \sum_{\substack{i+j=k\\ i,j\geq 1}} (\varphi\alpha)_{i,\ell_1} \otimes (\varphi\alpha)_{j,\ell_2}.\end{aligned}$$

Hence $\varphi\alpha$ is a coalgebra morphism.

We show that $\alpha\beta = \mathrm{id}$. Let $\varphi\colon TA \to B$ be a graded linear map of degree 0. Then given $k \geq 1$

$$\iota_k(\varphi\alpha\beta) = \iota_k(\varphi\alpha)\pi_1 = (\varphi\alpha)_{k,1} = \varphi_k = \iota_k\varphi,$$

hence $\varphi\alpha\beta = \varphi$. It follows that $\alpha\beta = \mathrm{id}$.
We show that β is injective. For this, suppose given coalgebra morphisms $f\colon TA \to TB$ and $g\colon TA \to TB$ with $f\beta = g\beta$, i.e. $f\pi_1 = g\pi_1$. we show that $\iota_k(f-g) = 0$ for all $k \geq 1$.
We use induction on k. For $k = 1$ we use that the first summand $A^{\otimes 1}$ is the kernel of Δ, thus $\iota_1\Delta = 0$, and obtain

$$\iota_1(f-g)\Delta = \iota_1\Delta(f\otimes f - g\otimes g) = 0.$$

It follows that $\iota_1(f-g) = \iota_1(f-g)\pi_1\iota_1 = \iota_1(f\pi_1 - g\pi_1)\iota_1 = 0$, cf. Remark 19.
Now let $k > 1$. Then, since by induction $\iota_i f = \iota_i g$ for $i < k$, we have using Remark 20

$$\begin{aligned}\iota_k(f-g)\Delta = \iota_k\Delta(f\otimes f - g\otimes g) &= \sum_{\substack{i+j=1\\ i,j\geq 1}} (\iota_i\otimes\iota_j)(f\otimes f - g\otimes g)\\ &= \sum_{\substack{i+j=1\\ i\,j\geq 1}} \iota_i f\otimes\iota_j f - \iota_i g\otimes\iota_j g\\ &= 0.\end{aligned}$$

Thus $\iota_k(f-g) = \iota_k(f-g)\pi_1\iota_1 = \iota_k(f\pi_1 - g\pi_1)\iota_1 = 0$, cf. Remark 19. Hence it follows by induction that β is injective.
Hence β is injective with $\alpha\beta = \mathrm{id}$, thus α and β are mutually inverse bijection.
Finally, for a coalgebra morphism $f\colon TA \to TB$ we have since $(f\beta)_i = (f\pi_1)i = \iota_i f\pi_1 = f_{i,1}$

$$f_{k,\ell} = (f\beta\alpha)_{k,\ell} = \sum_{\substack{i_1+\ldots+i_\ell=k\\ i_1,\ldots,i_\ell\geq 1}} (f\beta)_{i_1}\otimes\ldots\otimes(f\beta)_{i_\ell} = \sum_{\substack{i_1+\ldots+i_\ell=k\\ i_1,\ldots,i_\ell\geq 1}} f_{i_1,1}\otimes\ldots\otimes f_{i_\ell,1}$$

for $k,\ell \geq 1$.
(2) *We show that α is well-defined.* That is, given a graded linear map $\mu\colon TA \to A$ of degree 1 we show that $\mu\alpha$ is a coderivation.
It suffices to show that $\iota_k(\mu\alpha)\Delta(\pi_{\ell_1}\otimes\pi_{\ell_2}) = \iota_k\Delta(\mathrm{id}\otimes(\mu\alpha) + (\mu\alpha)\otimes\mathrm{id})(\pi_{\ell_1}\otimes\pi_{\ell_2})$ for all $k,\ell_1,\ell_2 \geq 1$. Using Remark 20 the left-hand side gives

$$\begin{aligned}\iota_k(\mu\alpha)\Delta(\pi_{\ell_1}\otimes\pi_{\ell_2}) &= \iota_k(\mu\alpha)\pi_{\ell_1+\ell_2}\\ &= (\mu\alpha)_{k,\ell_1+\ell_2},\end{aligned}$$

while the right-hand side gives

$$\begin{aligned}\iota_k\Delta(\mathrm{id}\otimes(\mu\alpha) + (\mu\alpha)\otimes\mathrm{id})(\pi_{\ell_1}\otimes\pi_{\ell_2}) &= \sum_{\substack{i+j=k\\ i,j\geq 1}} (\iota_i\otimes\iota_j)(\mathrm{id}\otimes(\mu\alpha) + (\mu\alpha)\otimes\mathrm{id})(\pi_{\ell_1}\otimes\pi_{\ell_2})\\ &= \sum_{\substack{i+j=k\\ i,j\geq 1}} \iota_i\pi_{\ell_1}\otimes\iota_j(\mu\alpha)\pi_{\ell_2} + \sum_{\substack{i+j=k\\ i,j\geq 1}} \iota_i(\mu\alpha)\pi_{\ell_1}\otimes\iota_j\pi_{\ell_2}\\ &= \mathrm{id}_{A^{\otimes\ell_1}}\otimes\iota_{k-\ell_1}(\mu\alpha)\pi_{\ell_2} + \iota_{k-\ell_2}(\mu\alpha)\pi_{\ell_1}\otimes\mathrm{id}_{A^{\otimes\ell_2}}\\ &= \mathrm{id}_A^{\otimes\ell_1}\otimes(\mu\alpha)_{k-\ell_1,\ell_2} + (\mu\alpha)_{k-\ell_2,\ell_1}\otimes\mathrm{id}_A^{\otimes\ell_2}\,.\end{aligned}$$

We obtain

$$
\begin{aligned}
(\mu\alpha)_{k,\ell_1+\ell_2} &= \sum_{\substack{r+s+t=k\\ r+1+t=\ell_1+\ell_2\\ r\geq 0,\, s\geq 1,\, t\geq 0}} \mathrm{id}_A^{\otimes r}\otimes \mu_s \otimes \mathrm{id}_A^{\otimes t}\\
&= \sum_{\substack{r+s+t=k\\ r+1+t=\ell_1+\ell_2\\ r\geq \ell_1,\, s\geq 1,\, t\geq 0}} \mathrm{id}_A^{\otimes r}\otimes \mu_s \otimes \mathrm{id}_A^{\otimes t} + \sum_{\substack{r+s+t=k\\ r+1+t=\ell_1+\ell_2\\ \ell_1-1\geq r\geq 0,\, s\geq 1,\, t\geq 0}} \mathrm{id}_A^{\otimes r}\otimes \mu_s \otimes \mathrm{id}_A^{\otimes t}\\
&= \sum_{\substack{r+s+t=k\\ r+1+t=\ell_1+\ell_2\\ r\geq \ell_1,\, s\geq 1,\, t\geq 0}} \mathrm{id}_A^{\otimes r}\otimes \mu_s \otimes \mathrm{id}_A^{\otimes t} + \sum_{\substack{r+s+t=k\\ r+1+t=\ell_1+\ell_2\\ r\geq 0,\, s\geq 1,\, t\geq \ell_2}} \mathrm{id}_A^{\otimes r}\otimes \mu_s \otimes \mathrm{id}_A^{\otimes t}\\
&= \sum_{\substack{u+s+t=k-\ell_1\\ u+1+t=\ell_2\\ u\geq 0,\, s\geq 1,\, t\geq 0}} \mathrm{id}_A^{\otimes(\ell_1+u)}\otimes \mu_s \otimes \mathrm{id}_A^{\otimes t} + \sum_{\substack{r+s+v=k-\ell_2\\ r+1+v=\ell_1\\ r\geq 0,\, s\geq 1,\, v\geq 0}} \mathrm{id}_A^{\otimes r}\otimes \mu_s \otimes \mathrm{id}_A^{\otimes(v+\ell_2)}\\
&= \mathrm{id}_A^{\otimes \ell_1}\otimes\Bigg(\sum_{\substack{u+s+t=k-\ell_1\\ u+1+t=\ell_2\\ u\geq 0,\, s\geq 1,\, t\geq 0}} \mathrm{id}_A^{\otimes u}\otimes \mu_s \otimes \mathrm{id}_A^{\otimes t}\Bigg) + \Bigg(\sum_{\substack{r+s+v=k-\ell_2\\ r+1+v=\ell_1\\ r\geq 0,\, s\geq 1,\, v\geq 0}} \mathrm{id}_A^{\otimes r}\otimes \mu_s \otimes \mathrm{id}_A^{\otimes v}\Bigg)\otimes \mathrm{id}_A^{\otimes \ell_2}\\
&= \mathrm{id}_A^{\otimes \ell_1}\otimes(\mu\alpha)_{k-\ell_1,\ell_2} + (\mu\alpha)_{k-\ell_2,\ell_1}\otimes \mathrm{id}_A^{\otimes \ell_2}\,.
\end{aligned}
$$

Hence $\mu\alpha$ is a coderivation.

We show that $\alpha\beta = \mathrm{id}$. Let $\mu\colon TA \to A$ be a graded linear map of degree 1. Given $k \geq 1$, we have

$$\iota_k(\mu\alpha\beta) = \iota_k(\mu\alpha)\pi_1 = (\mu\alpha)_{k,1} = \mu_k = \iota_k\mu,$$

hence $\mu\alpha\beta = \mu$, i.e. $\alpha\beta = \mathrm{id}$.

We show that β *is injective.* For this, we show that the kernel of β is trivial. Given a coderivation $m\colon TA \to TA$ with $m\beta = m\pi_1 = 0$, we show that $\iota_k m = 0$ for all $k \geq 1$. We use induction on k. For $k = 1$ we use that $\iota_1\Delta = 0$ since the first summand $A^{\otimes 1}$ is the kernel of Δ and obtain

$$\iota_1 m\Delta = \iota_1\Delta(\mathrm{id}\otimes m + m\otimes \mathrm{id}) = 0.$$

With Remark 19 we conclude that $\iota_1 m = \iota_1 m\pi_1\iota_1 = \iota_1(m\beta)\iota_1 = 0$. Now let $k > 1$. Then, since $\iota_i m = 0$ for $i < k$ by induction, we obtain using Remark 20

$$
\begin{aligned}
\iota_k m\Delta = \iota_k\Delta(\mathrm{id}\otimes m + m\otimes \mathrm{id}) &= \sum_{\substack{i+j=k\\ i,j\geq 1}} (\iota_i\otimes\iota_j)(\mathrm{id}\otimes m + m\otimes\mathrm{id})\\
&= \sum_{\substack{i+j=k\\ i\,j\geq 1}} (\iota_i\otimes\iota_j m + \iota_i m\otimes\iota_j) = 0.
\end{aligned}
$$

Again we conclude that $\iota_k m = \iota_k m\pi_1\iota_1 = \iota_k(m\beta)\iota_1 = 0$. Therefore it follows by induction that β is injective.

Hence β is an injective linear map with $\alpha\beta = \mathrm{id}$, hence α and β are mutually inverse isomorphisms.

Finally, for a coderivation $m\colon TA \to TA$ we have since $(m\beta)_i = (m\pi_1)_i = \iota_i m\pi_1 = m_{i,1}$

$$m_{k,\ell} = (m\beta\alpha)_{k,\ell} = \sum_{\substack{r+s+t=k\\ r+1+t=\ell\\ (r,s,t)\geq(0,1,0)}} \mathrm{id}^{\otimes r}\otimes(m\beta)_s\otimes\mathrm{id}^{\otimes t} = \sum_{\substack{r+s+t=k\\ r+1+t=\ell\\ (r,s,t)\geq(0,1,0)}} \mathrm{id}^{\otimes r}\otimes m_s\otimes\mathrm{id}^{\otimes t}$$

for $k, \ell \geq 1$. □

Lemma 23 *Let A and B be graded modules. For $k \geq 1$ let $T_{\leq k}A := \bigoplus_{1\leq j\leq k} A^{\otimes j} \subseteq TA$.*

(1) *Let $f\colon TA \to TB$ be a morphism of graded coalgebras. Then we have $f_{k,\ell} = 0$ for $1 \leq k < \ell$, i.e. we have $(T_{\leq k}A)f \subseteq T_{\leq k}B$.*

(2) *Let $m\colon TA \to TA$ be a coderivation. Then we have $m_{k,\ell} = 0$ for $1 \leq k < \ell$, i.e. we have $(T_{\leq k}A)m \subseteq T_{\leq k}A$.*

Proof. (1) By Lemma 22.(1) we have

$$f_{k,\ell} = \sum_{\substack{i_1+\ldots+i_\ell=k\\ i_1,\ldots,i_\ell\geq 1}} f_{i_1,1}\otimes\ldots\otimes f_{i_\ell,1}.$$

For $\ell > k$ the sum is empty, hence $f_{k,\ell} = 0$.
(2) By Lemma 22.(2) we have

$$m_{k,\ell} = \sum_{\substack{r+s+t=k\\ r+1+t=\ell\\ (r,s,t)\geq(0,1,0)}} \mathrm{id}^{\otimes r}\otimes m_{s,1}\otimes\mathrm{id}^{\otimes t}.$$

For $\ell > k$ the sum is empty, hence $m_{k,\ell} = 0$. □

Lemma 24 *Let A and B be graded modules.*

(1) *Suppose given a tuple $(\mu_k)_{k\geq 1}$ of graded linear maps $\mu_k\colon A^{\otimes k} \to A$ of degree 1. Let $\mu\colon TA \to A$ be the graded linear map with $\iota_k\mu = \mu_k$. By Lemma 22.(2), this defines a unique coderivation $m\colon TA \to TA$ on the tensor coalgebra with $m\pi_1 = \mu$.*

Then (TA, Δ, m) is a differential graded coalgebra, i.e. $m^2 = 0$, if and only if $(A, (\mu_k)_{k\geq 1})$ is an $\mathrm{A}_\infty^{[1]}$-algebra, i.e. the tuple $(\mu_k)_{k\geq 1}$ satisfies the Stasheff equation

$$0 = \sum_{\substack{r+s+t=k\\ (r,s,t)\geq(0,1,0)}} (\mathrm{id}^{\otimes r}\otimes\mu_s\otimes\mathrm{id}^{\otimes t})\mu_{r+1+t}$$

for $k \geq 1$, cf. also Definition 13.

(2) *Let $(A, (\mu_k)_{k\geq 1})$ and $(B, (\mu_k)_{k\geq 1})$ be $\mathrm{A}_\infty^{[1]}$-algebras. By (1), there are corresponding differential graded coalgebras (TA, Δ, m) and (TB, Δ, m).*

Suppose given graded linear maps $\varphi_k\colon A^{\otimes k} \to B$ of degree 0 for $k \geq 1$. Let $\varphi\colon TA \to B$ be the graded linear map with $\iota_k\varphi = \varphi_k$. By Lemma 22.(1), this defines a unique morphism of graded coalgebras $f\colon TA \to TB$ with $f\pi_1 = \varphi$.

Then f is a morphism of differential graded coalgebras, i.e. $fm = mf$, if and only if the tuple $(\varphi_k)_{k\geq 1}$ is a morphism of $\mathrm{A}_\infty^{[1]}$-algebras, i.e. it satisfies

$$\sum_{\substack{r+s+t=k\\(r,s,t)\geq(0,1,0)}} (\mathrm{id}^{\otimes r}\otimes \mu_s \otimes \mathrm{id}^{\otimes t})\varphi_{r+1+t} = \sum_{\ell=1}^{k} \sum_{\substack{i_1+\ldots+i_\ell=k\\ i_1,\ldots,i_\ell\geq 1}} (\varphi_{i_1}\otimes\ldots\otimes\varphi_{i_\ell})\mu_\ell$$

for $k\geq 1$, cf. also Definition 13.

Proof. (1) Let $k \geq 1$. By Lemma 23.(2) we have $(T_{\leq k}A)m \subseteq T_{\leq k}A$, hence we have $\iota_k m = \sum_{\ell=1}^{k} \iota_k m \pi_\ell \iota_\ell$. Using Lemma 22.(2) we obtain

$$\begin{aligned}\iota_k m^2 \pi_1 = \sum_{\ell=1}^{k} \iota_k m\pi_\ell\iota_\ell m &= \left(\sum_{\ell=1}^{k} \sum_{\substack{r+s+t=k\\ r+1+t=\ell\\(r,s,t)\geq(0,1,0)}} (\mathrm{id}^{\otimes r}\otimes \mu_s \otimes \mathrm{id}^{\otimes t})\iota_\ell\right) m\pi_1\\ &= \sum_{\substack{r+s+t=k\\(r,s,t)\geq(0,1,0)}} (\mathrm{id}^{\otimes r}\otimes \mu_s \otimes \mathrm{id}^{\otimes t})\mu_{r+1+t}\end{aligned}$$

Hence we have to show that $m^2 = 0$ if and only if $m^2\pi_1 = 0$. We only have to show the "if" part. Suppose that $m^2\pi_1 = 0$. We use induction on $k \geq 1$ to show that $\iota_k m^2 = 0$.
For $k = 1$ note that since $\iota_1\Delta = 0$ we have

$$\iota_1 m^2 \Delta = \iota_1\Delta(\mathrm{id}\otimes m + m\otimes \mathrm{id})(\mathrm{id}\otimes m + m \otimes \mathrm{id}) = 0,$$

hence using Remark 19 we have $\iota_1 m^2 = \iota_1 m^2 \pi_1 \iota_1 = 0$.
Now let $k > 1$. Using Remark 20, the Koszul sign rule and using that $\iota_i m^2 = 0$ for $i < k$ we obtain

$$\begin{aligned}\iota_k m^2\Delta &= \iota_k\Delta(\mathrm{id}\otimes m + m\otimes \mathrm{id})(\mathrm{id}\otimes m + m\otimes \mathrm{id})\\ &= \iota_k\Delta(\mathrm{id}\otimes m^2 + m\otimes\, m - m\otimes\, m + m^2\otimes \mathrm{id})\\ &= \sum_{\substack{i+j=k\\ i,j\geq 1}} (\iota_i\otimes\iota_j)(\mathrm{id}\otimes m^2 + m^2\otimes \mathrm{id})\\ &= \sum_{\substack{i+j=k\\ i,j\geq 1}} (\iota_i\otimes\, \iota_j m^2 + \iota_i m^2\otimes \iota_j)\\ &= 0.\end{aligned}$$

Again using Remark 19 gives $\iota_k m^2 = \iota_k m^2\pi_1\iota_1 = 0$.
Hence it follows by induction that $\iota_k m^2 = 0$ for all $k \geq 1$. Therefore $m^2 = 0$.
(2) Let $k \geq 1$. By Lemma 23.(1-2) we have $(T_{\leq k}A)f \subseteq T_{\leq k}A$ and $(T_{\leq k}A)m \subseteq T_{\leq k}A$, hence $\iota_k f = \sum_{\ell=1}^{k}\iota_k f\pi_\ell\iota_\ell$ and $\iota_k m = \sum_{\ell=1}^{k} \iota_k m\pi_\ell\iota_\ell$ for $k \geq 1$. Using Lemma 22.(1-2) we obtain

$$\begin{aligned}\iota_k m f\pi_1 = \sum_{\ell=1}^{k} \iota_k m\pi_\ell\iota_\ell f\pi_1 &= \left(\sum_{\ell=1}^{k} \sum_{\substack{r+s+t=k\\ r+1+t=\ell\\(r,s,t)\geq(0,1,0)}} (\mathrm{id}^{\otimes r}\otimes \mu_s \otimes \mathrm{id}^{\otimes t})\iota_\ell\right) f\pi_1\\ &= \sum_{\substack{r+s+t=k\\(r,s,t)\geq(0,1,0)}} (\mathrm{id}^{\otimes r}\otimes \mu_s \otimes \mathrm{id}^{\otimes t})\varphi_{r+1+t}\end{aligned}$$

and

$$\iota_k f m \pi_1 = \sum_{\ell=1}^{k} \iota_k f \pi_\ell \iota_\ell m \pi_1 = \left(\sum_{\ell=1}^{k} \sum_{\substack{i_1+\ldots+i_\ell=k \\ i_1,\ldots,i_\ell \geq 1}} (\varphi_{i_1} \otimes \ldots \otimes \varphi_{i_\ell}) \iota_\ell \right) m \pi_1$$

$$= \sum_{\ell=1}^{k} \sum_{\substack{i_1+\ldots+i_\ell=k \\ i_1,\ldots,i_\ell \geq 1}} (\varphi_{i_1} \otimes \ldots \otimes \varphi_{i_\ell}) \mu_\ell.$$

Hence we have to show that $fm = mf$ if and only if $fm\pi_1 = mf\pi_1$, i.e. we have to show that $fm - mf = 0$ if and only if $(fm - mf)\pi_1 = 0$. Of course, we only have to show the "if" part. For this, we use induction to show that $\iota_k(fm - mf) = 0$ for $k \geq 1$.
For $k = 1$ we use that $\iota_1 \Delta = 0$ since the first summand $A^{\otimes 1}$ is the kernel of Δ to obtain

$$\iota_1(fm - mf)\Delta = \iota_1 \Delta\big((f \otimes f)(\mathrm{id} \otimes m + m \otimes \mathrm{id}) - (\mathrm{id} \otimes m + m \otimes \mathrm{id})(f \otimes f)\big) = 0.$$

Hence $\iota_1(fm - mf) = \iota_1(fm - mf)\pi_1\iota_1 = 0$, cf. Remark 19.
Now let $k > 1$. Using Remark 20 and using that by induction $\iota_i(fm - mf) = 0$ for $i < k$, we have

$$\begin{aligned} \iota_k(fm - mf)\Delta &= \iota_k \Delta\big((f \otimes f)(\mathrm{id} \otimes m + m \otimes \mathrm{id}) - (\mathrm{id} \otimes m + m \otimes \mathrm{id})(f \otimes f)\big) \\ &= \sum_{\substack{i+j=k \\ i,j \geq 1}} (\iota_i \otimes \iota_j)(f \otimes fm + fm \otimes m - f \otimes mf - mf \otimes f) \\ &= \sum_{\substack{i+j=k \\ i,j \geq 1}} (\iota_i \otimes \iota_j)(f \otimes (fm - mf) + (fm - mf) \otimes f) \\ &= 0. \end{aligned}$$

Again using Remark 19 we conclude that $\iota_k(fm - mf) = \iota_k(fm - mf)\pi_1\iota_1 = 0$.
Hence it follows by induction that $\iota_k(fm - mf) = 0$ for $k \geq 1$. Therefore $fm = mf$. □

We remark that the proof of Lemma 24.(2) can be simplified using the results of §2.1 below. In fact, $fm - mf$ is an (f, f)-coderivation in the sense of Definition 34. This follows for example using Lemma 36 since m is an $(\mathrm{id}, \mathrm{id})$-coderivation. The assertion $fm = mf$ if and only if $fm\pi_1 = mf\pi_1$ then follows immediately from Lemma 37.

Lemma 25 *Let A and B be graded modules and suppose give a morphism of graded coalgebras $f\colon TA \to TB$ between their tensor coalgebras.*
If $f_{1,1}$ is a split monomorphism, then f is injective.

Proof. By Lemma 23 $(T_{\leq k}A)f \subseteq T_{\leq k}B$ for all $k \geq 1$, hence we can define the restriction

$$f_{\leq k} := f|_{T_{\leq k}A}^{T_{\leq k}B} \colon T_{\leq k}A \to T_{\leq k}B.$$

By Lemma 22.(1), we have $f_{k,k} = (f_{1,1})^{\otimes k}$, hence $f_{k,k}$ is a split monomorphism for $k \geq 1$.

We *claim* that $f_{\leq k}$ is an injective graded linear map for $k \geq 1$. We use induction on k. Since $f_{\leq 1} = f_{1,1}$, the case $k = 1$ is our assumption. Now let $k \geq 1$. Consider the following morphism of short exact sequences of graded linear maps.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_{\leq k}A & \xrightarrow{i^A_{\leq k}} & T_{\leq k+1}A & \xrightarrow{p^A_{k+1}} & A^{\otimes(k+1)} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_{\leq k}} & & \downarrow{\scriptstyle f_{\leq k+1}} & & \downarrow{\scriptstyle f_{k+1,k+1}} & & \\
0 & \longrightarrow & T_{\leq k}B & \xrightarrow{i^B_{\leq k}} & T_{\leq k+1}B & \xrightarrow{p^B_{k+1}} & B^{\otimes(k+1)} & \longrightarrow & 0
\end{array}$$

Here $i^A_{\leq k}$ and $i^B_{\leq k}$ are inclusions of direct summands and p^A_{k+1} and p^B_{k+1} are projections onto direct summands. By induction, $f_{\leq k}$ is injective. We also know that $f_{k+1,k+1}$ is injective. Adding the kernels of the vertical maps to the above diagram gives the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \ker(f_{\leq k+1}) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_{\leq k}A & \xrightarrow{i^A_{\leq k}} & T_{\leq k+1}A & \xrightarrow{p^A_{k+1}} & A^{\otimes(k+1)} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_{\leq k}} & & \downarrow{\scriptstyle f_{\leq k+1}} & & \downarrow{\scriptstyle f_{k+1,k+1}} & & \\
0 & \longrightarrow & T_{\leq k}B & \xrightarrow{i^B_{\leq k}} & T_{\leq k+1}B & \xrightarrow{p^B_{k+1}} & B^{\otimes(k+1)} & \longrightarrow & 0
\end{array}$$

But then $\ker(f_{\leq k+1}) = 0$, hence $f_{\leq k+1}$ is also injective. Therefore the *claim* follows by induction.

Suppose given $z \in \mathrm{Mor}(\mathcal{Z})$ and $a_1, a_2 \in (TA)^z$ with $a_1 f^z = a_2 f^z$. Since TA is an infinite direct sum we can find a $k \geq 1$ such that $a_1, a_2 \in (T_{\leq k}A)^z$. But since $f_{\leq k}$ is injective, it follows that $a_1 = a_2$. Therefore f is an injective graded linear map. □

Lemma 26 *Let A and B be graded modules and suppose given a morphism of graded coalgebras $f\colon TA \to TB$ between their tensor coalgebras.*
Then f is an isomorphism of graded coalgebras if and only if the component $f_{1,1}\colon A \to B$ is an isomorphism of graded modules.

Proof. Suppose that f is an isomorphism of graded coalgebras. Then there is a morphism of graded coalgebras $g\colon TB \to TA$ such that $fg = \mathrm{id}_{TA}$ and $gf = \mathrm{id}_{TB}$. By Lemma 23, the coalgebra morphisms f and g satisfy $(T_{\leq 1}A)f \subseteq T_{\leq 1}B$ and $(T_{\leq 1}B)g \subseteq T_{\leq 1}A$. Hence we have $\iota_1 f = \iota_1 f \pi_1 \iota_1 = f_{1,1}\iota_1$ and $\iota_1 g = \iota_1 g \pi_1 \iota_1 = g_{1,1}\iota_1$. It follows that

$$f_{1,1}g_{1,1} = f_{1,1}\iota_1 g \pi_1 = \iota_1 f g \pi_1 = \mathrm{id}_A \quad \text{and} \quad g_{1,1}f_{1,1} = g_{1,1}\iota_1 f \pi_1 = \iota_1 g f \pi_1 = \mathrm{id}_B\,.$$

Thus $f_{1,1}$ is an isomorphism of graded modules.

Conversely, suppose that $f_{1,1}\colon A \to B$ is an isomorphism of graded modules. By Lemma 23 $(T_{\leq k})f \subseteq T_{\leq k}B$ for all $k \geq 1$, hence we can define the restriction

$$f_{\leq k} := f|^{T_{\leq k}B}_{T_{\leq k}A}\colon T_{\leq k}A \to T_{\leq k}B.$$

By Lemma 22.(1), we have $f_{k,k} = (f_{1,1})^{\otimes k}$, hence $f_{k,k}$ is an isomorphism for all $k \geq 1$.

We *claim* that $f_{\leq k}$ is an isomorphism of graded modules for all $k \geq 1$. We use induction on k. Since $f_{\leq 1} = f_{1,1}$, the case $k = 1$ is our assumption. Now let $k \geq 1$. Consider the following morphism of short exact sequences of graded linear maps.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_{\leq k}A & \xrightarrow{i^A_{\leq k}} & T_{\leq k+1}A & \xrightarrow{p^A_{k+1}} & A^{\otimes(k+1)} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_{\leq k}} & & \downarrow{\scriptstyle f_{\leq k+1}} & & \downarrow{\scriptstyle f_{k+1,k+1}} & & \\
0 & \longrightarrow & T_{\leq k}B & \xrightarrow{i^B_{\leq k}} & T_{\leq k+1}B & \xrightarrow{p^B_{k+1}} & B^{\otimes(k+1)} & \longrightarrow & 0
\end{array}$$

Here $i^A_{\leq k}$ and $i^B_{\leq k}$ are inclusions of direct summands and p^A_{k+1} and p^B_{k+1} are projections onto direct summands. By the inductive hypothesis, $f_{\leq k}$ is an isomorphism and the morphism $f_{k+1,k+1} = (f_{1,1})^{\otimes(k+1)}$ is an isomorphism since $f_{1,1}$ is by assumption. Hence by the five lemma in abelian categories also $f_{\leq k+1}$ is an isomorphism. Therefore the *claim* follows by induction.

To show that f is an isomorphism we show that f is bijective, i.e. we show that f^z is bijective for all $z \in \mathrm{Mor}(\mathcal{Z})$. Given $b \in (TB)^z$ there is a $k \geq 1$ such that $b \in (T_{\leq k}B)^z$. Since $f_{\leq k}$ is surjective, we can find a preimage of b under f. For injectivity, let $a_1, a_2 \in (TA)^z$ with $a_1 f^z = a_2 f^z$. Since TA is an infinite direct sum we can find a $k \geq 1$ such that $a_1, a_2 \in (T_{\leq k}A)^z$. But since $f_{\leq k}$ is injective, it follows that $a_1 = a_2$.

Hence f is a bijective map of graded modules, hence an isomorphism. Let g be its inverse. Then

$$g\Delta = g\Delta(f \otimes f)(g \otimes g) = gf\Delta(g \otimes g) = \Delta(g \otimes g),$$

therefore g is a morphism of graded coalgebras. We conclude that f is an isomorphism of graded coalgebras. □

1.3.3 The Bar construction

Let $A := (A, (\mathsf{m}_k)_{k\geq 1})$ and $B := (B, (\mathsf{m}_k)_{k\geq 1})$ be A_∞-algebras and let $(A^{[1]}, (\mu_k)_{k\geq 1})$ and $(B^{[1]}, (\mu_k)_{k\geq 1})$ be the corresponding $\mathrm{A}_\infty^{[1]}$-algebras, cf. Definitions 13, 14 and Remark 15.(1). Let $\mathrm{A}_\infty\text{-}\mathsf{alg}(A, B)$ be the set of A_∞-morphisms from A to B.

Lemma 27

(1) *There is a uniquely determined differential m on the tensor coalgebra $(TA^{[1]}, \Delta)$ with $m_{k,1} = \mu_k$ for $k \geq 1$ such that* $\mathrm{Bar}\,A := (TA^{[1]}, \Delta, m)$ *is a differential graded coalgebra.*

(2) *There is a bijection*

$$\begin{array}{rccc}
\mathrm{Bar}\colon & \mathrm{A}_\infty\text{-}\mathsf{alg}(A, B) & \longrightarrow & \mathsf{dgCoalg}(\mathrm{Bar}\,A, \mathrm{Bar}\,B) \\
 & f & \longmapsto & \mathrm{Bar}\,f.
\end{array}$$

For an A_∞-morphism f the differential graded coalgebra morphism $\mathrm{Bar}\,f\colon TA^{[1]} \to TB^{[1]}$ is constructed as follows. Let $\varphi\colon (A^{[1]}, (\mu_k)_{k\geq 1}) \to (B^{[1]}, (\mu_k)_{k\geq 1})$ be the $\mathrm{A}_\infty^{[1]}$-morphism corresponding to f. Then $\mathrm{Bar}\,f$ is the uniquely determined morphism of differential graded coalgebras with $(\mathrm{Bar}\,f)_{k,1} = \varphi_k$ for $k \geq 1$, cf. Lemma 22.(1).

Proof. (1) By Lemma 22.(2) there is a unique coderivation m on $TA^{[1]}$ with $m_{k,1} = \mu_k$. By Lemma 24.(1) the coderivation m is a differential, since $(\mu_k)_{k\geq 1}$ satisfies the Stasheff equations.
(2) Let $f \in \mathrm{A}_\infty\text{-}\mathsf{alg}(A, B)$ be a A_∞-algebra morphism. By Remark 15 there is a bijection between A_∞-algebra morphism from $(A, (\mathsf{m}_k)_{k\geq 1})$ to $(B, (\mathsf{m}_k)_{k\geq 1})$ and $\mathrm{A}_\infty^{[1]}$-algebra morphism from $(A^{[1]}, (\mu_k)_{k\geq 1})$ to $(B^{[1]}, (\mu_k)_{k\geq 1})$. Let φ be the $\mathrm{A}_\infty^{[1]}$-algebra morphism corresponding to f under this bijection.
By Lemma 22.(1) there is a bijection between graded linear maps $TA^{[1]} \to B^{[1]}$, i.e. tuples of maps $(A^{[1]})^{\otimes k} \to B^{[1]}$ for $k \geq 1$, and coalgebra morphisms $TA^{[1]} \to TB^{[1]}$. By Lemma 24.(2) this bijection restricts to a bijection between $\mathrm{A}^{[1]}$-algebra morphisms from $(A^{[1]}, (\mu_k)_{k\geq 1})$ to $(B^{[1]}, (\mu_k)_{k\geq 1})$ and differential graded coalgebra morphisms from $\operatorname{Bar} A$ to $\operatorname{Bar} B$. □

Definition 28 We define the category $\mathrm{A}_\infty\text{-}\mathsf{alg}$ of A_∞-algebras that has as objects A_∞-algebras $A = (A, (m_k)_{k\geq 1})$ and morphisms of A_∞-algebras as morphisms. Composition is defined by transport of structure such that

$$\begin{array}{rrcl} \operatorname{Bar}\colon & \mathrm{A}_\infty\text{-}\mathsf{alg} & \longrightarrow & \mathsf{dgCoalg} \\ & A & \longmapsto & \operatorname{Bar} A \\ & (f\colon A \to B) & \longmapsto & (\operatorname{Bar} f\colon \operatorname{Bar} A \to \operatorname{Bar} B) \end{array}$$

defines a full and faithful functor, cf. Lemma 27.

Definition 29 Let $\mathsf{dtCoalg}$ be the full subcategory of $\mathsf{dgCoalg}$ consisting of those differential graded coalgebras whose underlying graded coalgebra is a tensor coalgebra over some graded module.
We will call an object in $\mathsf{dtCoalg}$ a *differential graded tensor coalgebra.*
Note that the Bar functor from Definition 28 restricts to an equivalence of categories

$$\operatorname{Bar}\colon \quad \mathrm{A}_\infty\text{-}\mathsf{alg} \xrightarrow{\sim} \mathsf{dtCoalg} \subseteq \mathsf{dgCoalg}\,.$$

1.3.4 Attaching a counit

In Definition 16, we defined the categories of graded coalgebras $\mathsf{grCoalg}$ and counital graded coalgebras $\mathsf{grCoalg}_*$. There is a forgetful functor $V\colon \mathsf{grCoalg}_* \to \mathsf{grCoalg}$ that sends a counital graded coalgebra (C, Δ, ε) to the graded coalgebra (C, Δ) and each morphism to itself.
We construct a right adjoint of V, i.e. a functor $E\colon \mathsf{grCoalg} \to \mathsf{grCoalg}_*$ that "attaches" a counit to a graded coalgebra.

Lemma 30

(1) *Given a graded coalgebra $C = (C, \Delta)$, the graded module $\hat{C} := \dot{R} \oplus C$ is a counital graded coalgebra with comultiplication and counit given as follows.*

$$\begin{array}{rrcl} \hat{\Delta}\colon & \dot{R} \oplus C & \longrightarrow & (\dot{R} \oplus C) \otimes (\dot{R} \oplus C) \\ \hat{\Delta}^z\colon & (r, c) & \longmapsto & (r, 0) \otimes (1, 0) + (1, 0) \otimes (0, c) + (0, c) \otimes (1, 0) + c\Delta^z(\iota \otimes \iota)^z \end{array}$$

$$\begin{array}{rrcl} \hat{\varepsilon}\colon & \dot{R} \oplus C & \longrightarrow & \dot{R} \\ \hat{\varepsilon}^z\colon & (r, c) & \longmapsto & r \end{array}$$

Here, $\iota\colon C \to \dot{R} \oplus C$ *denotes the graded linear map of degree* 0 *given by inclusion of the direct summand.*

Note that for $z\colon x \to y$ *in* $\mathcal{Z}$ *and the summand* $(1,0) \otimes (0,c)$ *in the definition of* $\hat{\Delta}^z$ *above we have* $(1,0) \in (\dot{R} \oplus C)^{\mathrm{id}_x}$ *and* $(0,c) \in (\dot{R} \oplus C)^z$*. For the summand* $(0,c) \otimes (1,0)$ *we have* $(0,c) \in (\dot{R} \oplus C)^z$ *and* $(1,0) \in (\dot{R} \oplus C)^{\mathrm{id}_y}$*.*

(2) *Given a morphism* $f\colon C \to D$ *between graded coalgebras* $C = (C, \Delta)$ *and* $D = (D, \Delta)$*, the graded linear map*

$$\begin{array}{rrcl} \hat{f}\colon & \dot{R} \oplus C & \longrightarrow & \dot{R} \oplus C \\ \hat{f}^z\colon & (r,c) & \longmapsto & (r, cf^z) \end{array}$$

is a morphism of counital graded coalgebras.

(3) *We have the functor*

$$\begin{array}{rrcl} E\colon & \mathsf{grCoalg} & \longrightarrow & \mathsf{grCoalg}_* \\ & C & \longmapsto & \hat{C} \\ & f & \longmapsto & \hat{f}. \end{array}$$

Proof. (1) We have to show coassociativity of $\hat{\Delta}$ and the counit property of $\hat{\varepsilon}$. For coassociativity of $\hat{\Delta}$, we *claim* that the following equation holds for $z \in \mathrm{Mor}(\mathcal{Z})$ and $c \in C^z$.

$$c\Delta^z(\iota \otimes \iota)^z(\mathrm{id} \otimes \hat{\Delta})^z + (1,0) \otimes c\Delta^z(\iota \otimes \iota)^z = c\Delta^z(\iota \otimes \iota)^z(\hat{\Delta} \otimes \mathrm{id})^z + c\Delta^z(\iota \otimes \iota)^z \otimes (1,0) \tag{$*$}$$

To show the claim, let $c\Delta^z = \sum_{i=1}^n c_i \otimes c_i'$ for elements $c_i \in C^{z_i}$ and $c_i' \in C^{z_i'}$ for $z_i, z_i' \in \mathrm{Mor}(\mathcal{Z})$ with $z_i z_i' = z$. We calculate.

$$\begin{aligned} & c\Delta^z(\iota \otimes \iota)^z(\mathrm{id} \otimes \hat{\Delta})^z + (1,0) \otimes c\Delta^z(\iota \otimes \iota)^z \\ &\quad = \sum_{i=1}^n (c_i \otimes c_i')(\iota \otimes \iota)^z(\mathrm{id} \otimes \hat{\Delta})^z + \sum_{i=1}^n (1,0) \otimes (c_i \otimes c_i')(\iota \otimes \iota)^z \\ &\quad = \sum_{i=1}^n (0,c_i) \otimes (0,c_i')\hat{\Delta}^{z_i'} + \sum_{i=1}^n (1,0) \otimes (0,c_i) \otimes (0,c_i') \\ &\quad = \sum_{i=1}^n (0,c_i) \otimes (1,0) \otimes (0,c_i') + \sum_{i=1}^n (0,c_i) \otimes (0,c_i') \otimes (1,0) \\ &\qquad + \sum_{i=1}^n (0,c_i) \otimes c_i'\Delta^{z_i'}(\iota \otimes \iota)^{z_i'} + \sum_{i=1}^n (1,0) \otimes (0,c_i) \otimes (0,c_i') \end{aligned}$$

Similarly, we obtain

$$
\begin{aligned}
&c\Delta^z(\iota\otimes\iota)^z(\hat{\Delta}\otimes\mathrm{id})^z + c\Delta^z(\iota\otimes\iota)^z\otimes(1,0)\\
&\qquad = \sum_{i=1}^n (c_i\otimes c_i')(\iota\otimes\iota)^z(\hat{\Delta}\otimes\mathrm{id})^z + \sum_{i=1}^n (c_i\otimes c_i')(\iota\otimes\iota)^z\otimes(1,0)\\
&\qquad = \sum_{i=1}^n (0,c_i)\hat{\Delta}^{z_i}\otimes(0,c_i') + \sum_{i=1}^n (0,c_i)\otimes(0,c_i')\otimes(1,0)\\
&\qquad = \sum_{i=1}^n (1,0)\otimes(0,c_i)\otimes(0,c_i') + \sum_{i=1}^n (0,c_i)\otimes(1,0)\otimes(0,c_i')\\
&\qquad\quad + \sum_{i=1}^n c_i\Delta^{z_i}(\iota\otimes\iota)^{z_i}\otimes(0,c_i') + \sum_{i=1}^n (0,c_i)\otimes(0,c_i')\otimes(1,0).
\end{aligned}
$$

Finally, we have

$$
\sum_{i=1}^n (0,c_i)\otimes c_i'\Delta^{z_i'}(\iota\otimes\iota)^{z_i'} = \sum_{i=1}^n (c_i\otimes c_i'\Delta^{z_i'})(\iota\otimes\iota\otimes\iota)^z = c\Delta^z(\mathrm{id}\otimes\Delta)^z(\iota\otimes\iota\otimes\iota)^z
$$

and

$$
\sum_{i=1}^n c_i\Delta^{z_i}(\iota\otimes\iota)^{z_i}\otimes(0,c_i') = \sum_{i=1}^n (c_i\Delta^{z_i}\otimes c_i')(\iota\otimes\iota\otimes\iota)^z = c\Delta^z(\Delta\otimes\mathrm{id})^z(\iota\otimes\iota\otimes\iota)^z,
$$

thus the *claim* $(*)$ follows using coassociativity of Δ.
We are now able to show coassociativity of $\hat{\Delta}$. Let $z\colon x\to y$ be a morphism in $\mathcal{Z}$ and let $(r,c)\in(\dot{R}\oplus C)^z$. We calculate.

$$
\begin{aligned}
(r,c)\hat{\Delta}^z(\mathrm{id}\otimes\hat{\Delta})^z &= \Big((r,0)\otimes(1,0)+(1,0)\otimes(0,c)+(0,c)\otimes(1,0)+c\Delta^z(\iota\otimes\iota)^z\Big)(\mathrm{id}\otimes\hat{\Delta})^z\\
&= (r,0)\otimes(1,0)\hat{\Delta}^z + (1,0)\otimes(0,c)\hat{\Delta}^z + (0,c)\otimes(1,0)\hat{\Delta}^z + c\Delta^z(\iota\otimes\iota)^z(\mathrm{id}\otimes\hat{\Delta})^z\\
&= (r,0)\otimes(1,0)\otimes(1,0) + (1,0)\otimes\Big((1,0)\otimes(0,c)+(0,c)\otimes(1,0)+c\Delta^z(\iota\otimes\iota)^z\Big)\\
&\quad + (0,c)\otimes(1,0)\otimes(1,0) + c\Delta^z(\iota\otimes\iota)^z(\mathrm{id}\otimes\hat{\Delta})^z\\
&= (r,0)\otimes(1,0)\otimes(1,0)\\
&\quad + (1,0)\otimes(1,0)\otimes(0,c) + (1,0)\otimes(0,c)\otimes(1,0) + (0,c)\otimes(1,0)\otimes(1,0)\\
&\quad + c\Delta^z(\iota\otimes\iota)^z(\mathrm{id}\otimes\hat{\Delta})^z + (1,0)\otimes c\Delta^z(\iota\otimes\iota)^z\\
(r,c)\hat{\Delta}^z(\hat{\Delta}\otimes\mathrm{id})^z &= \Big((r,0)\otimes(1,0)+(1,0)\otimes(0,c)+(0,c)\otimes(1,0)+c\Delta^z(\iota\otimes\iota)^z\Big)(\hat{\Delta}\otimes\mathrm{id})^z\\
&= (r,0)\hat{\Delta}^z\otimes(1,0) + (1,0)\hat{\Delta}^z\otimes(0,c) + (0,c)\hat{\Delta}^z\otimes(1,0) + c\Delta^z(\iota\otimes\iota)^z(\hat{\Delta}\otimes\mathrm{id})^z\\
&= (r,0)\otimes(1,0)\otimes(1,0) + (1,0)\otimes(1,0)\otimes(0,c)\\
&\quad + \Big((1,0)\otimes(0,c)+(0,c)\otimes(1,0)+c\Delta^z(\iota\otimes\iota)^z\Big)\otimes(1,0) + c\Delta^z(\iota\otimes\iota)^z(\hat{\Delta}\otimes\mathrm{id})^z\\
&= (r,0)\otimes(1,0)\otimes(1,0)\\
&\quad + (1,0)\otimes(1,0)\otimes(0,c) + (1,0)\otimes(0,c)\otimes(1,0) + (0,c)\otimes(1,0)\otimes(1,0)\\
&\quad + c\Delta^z(\iota\otimes\iota)^z(\hat{\Delta}\otimes\mathrm{id})^z + c\Delta^z(\iota\otimes\iota)^z\otimes(1,0).
\end{aligned}
$$

Thus coassociativity $\hat{\Delta}(\mathrm{id}\otimes\hat{\Delta}) = \hat{\Delta}(\hat{\Delta}\otimes\mathrm{id})$ follows from $(*)$.
It remains to show that $\hat{\varepsilon}$ is a counit, i.e. that $\hat{\Delta}(\mathrm{id}\otimes\hat{\varepsilon}) = \mathrm{id} = \hat{\Delta}(\hat{\varepsilon}\otimes\mathrm{id})$. Note that by definition of $\hat{\varepsilon}$ we have $\iota\hat{\varepsilon} = 0$. Note that we identify along the tensor unit isomorphisms, cf. Remark 8. We calculate.

$$\begin{aligned}
(r,c)\hat{\Delta}(\mathrm{id}\otimes\hat{\varepsilon}) &= \big((r,0)\otimes(1,0) + (1,0)\otimes(0,c) + (0,c)\otimes(1,0) + c\Delta^z(\iota\otimes\iota)^z\big)(\mathrm{id}\otimes\hat{\varepsilon})\\
&= (r,0)\otimes 1 + (0,c)\otimes 1\\
&= (r,c)\\
(r,c)\hat{\Delta}(\hat{\varepsilon}\otimes\mathrm{id}) &= \big((r,0)\otimes(1,0) + (1,0)\otimes(0,c) + (0,c)\otimes(1,0) + c\Delta^z(\iota\otimes\iota)^z\big)(\hat{\varepsilon}\otimes\mathrm{id})\\
&= r\otimes(1,0) + 1\otimes(0,c)\\
&= (r,c)
\end{aligned}$$

Hence $\hat{\varepsilon}$ is a counit. It follows that $(\hat{C},\hat{\Delta},\hat{\varepsilon})$ is a counital coalgebra.
(2) We have to show $\hat{f}\hat{\Delta} = \hat{\Delta}(\hat{f}\otimes\hat{f})$ and $\hat{f}\hat{\varepsilon} = \hat{\varepsilon}$. Let $z\in\mathrm{Mor}(\mathcal{Z})$ and $(r,c)\in\hat{C}^z$. Note that $\iota\hat{f} = f\iota$. We calculate.

$$\begin{aligned}
(r,c)\hat{f}^z\hat{\Delta}^z &= (r,cf^z)\hat{\Delta}^z\\
&= (r,0)\otimes(1,0) + (1,0)\otimes(0,cf^z) + (0,cf^z)\otimes(1,0) + cf^z\Delta^z(\iota\otimes\iota)^z\\
(r,c)\hat{\Delta}^z(\hat{f}\otimes\hat{f})^z &= \Big((r,0)\otimes(1,0) + (1,0)\otimes(0,c) + (0,c)\otimes(1,0) + c\Delta^z(\iota\otimes\iota)^z\Big)(\hat{f}\otimes\hat{f})^z\\
&= (r,0)\otimes(1,0) + (1,0)\otimes(0,cf^z) + (0,cf^z)\otimes(1,0) + c\Delta^z(f\otimes f)^z(\iota\otimes\iota)^z
\end{aligned}$$

Hence $\hat{f}$ is a coalgebra morphism since f is a coalgebra morphism, i.e. $f\Delta = \Delta(f\otimes f)$. Moreover, we have

$$(r,c)\hat{f}^z\hat{\varepsilon}^z = (r,cf^z)\hat{\varepsilon}^z = r = (r,c)\hat{\varepsilon}^z.$$

Hence $\hat{f}\hat{\varepsilon} = \hat{\varepsilon}$ and the assertion follows.
(3) By (1) and (2) the maps on objects and morphisms are well-defined. It remains to show that $E\,\mathrm{id} = \mathrm{id}$ and $E(fg) = (Ef)(Eg)$ for coalgebra morphisms $f\colon C\to D$ and $g\colon D\to B$. Let $z\in\mathrm{Mor}(\mathcal{C})$ and $(r,c)\in(EC)^z = \hat{C}^z$. Then

$$(r,c)(E\,\mathrm{id})^z = (r,c\,\mathrm{id}^z) = (r,c),$$

hence $E\,\mathrm{id} = \mathrm{id}$. Moreover, we have

$$(r,c)\big(E(fg)\big)^z = (r,c(fg)^z) = (r,cf^zg^z) = (r,c)(Ef)^z(Eg)^z,$$

hence $E(fg) = (Ef)(Eg)$. It follows that E is a functor. □

Lemma 31

(1) *Given a graded coalgebra $C = (C,\Delta)$, the graded linear map*

$$\begin{array}{rrcl}
\rho_C\colon & \hat{C} = \dot{R}\oplus C & \longrightarrow & C\\
\rho_C^z\colon & (r,c) & \longmapsto & c
\end{array}$$

is a morphism of graded coalgebras. Moreover, the morphisms ρ_C define a natural transformation $\rho = (\rho_C)_C\colon VE\to\mathrm{id}$.

(2) *Suppose we are given a counital graded coalgebra $C = (C, \Delta, \varepsilon)$ and a graded coalgebra $D = (D, \Delta)$. Given a morphism of graded coalgebras $f\colon C \to D$, there is a unique morphism of counital graded coalgebras $\bar{f}\colon C \to \hat{D}$ such that $\bar{f}\rho_D = f$.*

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ & \overset{\exists!\bar{f}}{\searrow} & \uparrow \rho_D \\ & & \hat{D} \end{array}$$

(3) *The forgetful functor V is a left adjoint to the functor E.*

$$\mathsf{grCoalg}_* \underset{E}{\overset{V}{\rightleftarrows}} \mathsf{grCoalg} \quad (V \dashv E)$$

Proof. (1) Let $z \in \mathrm{Mor}(\mathcal{Z})$ and $(r, c) \in (\dot{R} \oplus C)^z$. Note that $\iota\rho_C = \mathrm{id}$. We calculate.

$$\begin{aligned} (r,c)\hat{\Delta}^z(\rho_C \otimes \rho_C)^z &= \big((r,0) \otimes (1,0) + (1,0) \otimes (0,c) + (0,c) \otimes (1,0) + c\Delta^z(\iota \otimes \iota)^z\big)(\rho_C \otimes \rho_C)^z \\ &= c\Delta^z \\ &= (r,c)\rho_C^z\Delta^z \end{aligned}$$

Hence ρ_C is a morphism of graded coalgebras.
For naturality of ρ, let $g\colon C \to D$ be a morphism of graded coalgebras. We have to show that the following diagram commutes.

$$\begin{array}{ccc} \hat{C} = VEC & \xrightarrow{\rho_C} & C \\ {\scriptstyle \hat{g} = VEg}\downarrow & & \downarrow{\scriptstyle g} \\ \hat{D} = VED & \xrightarrow{\rho_D} & D \end{array}$$

Given $z \in \mathrm{Mor}(\mathcal{Z})$ and $(r, c) \in \hat{C}^z = (\dot{R} \oplus C)^z$ we have

$$(r,c)\rho_C^z g^z = cg^z \quad \text{and} \quad (r,c)\hat{g}^z\rho_D^z = (r, cg^z)\rho_D^z = cg^z.$$

It follows that $\rho_C g = \hat{g}\rho_D$. Therefore $\rho\colon VE \to \mathrm{id}$ is a natural transformation.
(2) *Uniqueness.* Since $\bar{f}$ has to satisfy both $\bar{f}\hat{\varepsilon} = \varepsilon$ and $\bar{f}\rho_D = f$, we necessarily have $c\bar{f}^z = (c\varepsilon^z, cf^z)$ for $z \in \mathrm{Mor}(\mathcal{Z})$ and $c \in C^z$. It follows that $\bar{f}$ is uniquely determined.
Existence. We define

$$\begin{array}{rccc} \bar{f}\colon & C & \longrightarrow & \hat{D} = \dot{R} \oplus D \\ \bar{f}^z\colon & c & \longmapsto & (c\varepsilon^z, cf^z). \end{array}$$

We have to show that $\bar{f}$ is a morphism of counital graded coalgebras. Let $z \in \mathrm{Mor}(\mathcal{Z})$ and $c \in C^z$. Write $c\Delta^z = \sum_{i=1}^n c_i \otimes c_i'$ where $c_i \in C^{z_i}$ and $c_i' \in C^{z_i'}$ are elements with $z_i, z_i' \in \mathrm{Mor}(\mathcal{Z})$

such that $z_i z_i' = z$. We calculate.

$$
\begin{aligned}
c\Delta^z(\bar f\otimes \bar f)^z &= \sum_{i=1}^n (c_i\otimes c_i')(\bar f\otimes \bar f)^z\\
&= \sum_{i=1}^n c_i \bar f^{z_i}\otimes c_i'\bar f^{z_i'}\\
&= \sum_{i=1}^n (c_i\varepsilon^{z_i}, c_i f^{z_i})\otimes (c_i'\varepsilon^{z_i'}, c_i' f^{z_i'})\\
&= \sum_{i=1}^n (c_i\varepsilon^{z_i}\cdot c_i'\varepsilon^{z_i'},0)\otimes(1,0) + \sum_{i=1}^n (1,0)\otimes(0, c_i\varepsilon^{z_i}\cdot c_i' f^{z_i'})\\
&\quad + \sum_{i=1}^n (0, c_i f^{z_i}\cdot c_i'\varepsilon^{z_i'})\otimes(1,0) + \sum_{i=1}^n (0, c_i f^{z_i})\otimes(0, c_i' f^{z_i'})\\
c\bar f\hat\Delta^z &= (c\varepsilon^z, cf^z)\hat\Delta^z\\
&= (c\varepsilon^z,0)\otimes(1,0) + (1,0)\otimes(0,cf^z) + (0,cf^z)\otimes(1,0) + cf^z\Delta^z(\iota\otimes\iota)^z
\end{aligned}
$$

Using the counit property $\Delta(\mathrm{id}\otimes\varepsilon) = \mathrm{id} = \Delta(\varepsilon\otimes\mathrm{id})$ we obtain

$$
\begin{aligned}
\sum_{i=1}^n c_i\varepsilon^{z_i}\cdot c_i'\varepsilon^{z_i'} &= \sum_{i=1}^n (c_i\varepsilon^{z_i}\cdot c_i')\varepsilon^z = \sum_{i=1}^n (c_i\varepsilon^{z_i}\otimes c_i')\varepsilon^z\\
&= \sum_{i=1}^n (c_i\otimes c_i')(\varepsilon\otimes \mathrm{id})^z\varepsilon^z = c\Delta^z(\varepsilon\otimes\mathrm{id})^z\varepsilon^z = c\varepsilon^z\\
\sum_{i=1}^n c_i\varepsilon^{z_i}\cdot c_i' f^{z_i'} &= \sum_{i=1}^n (c_i\varepsilon^{z_i}\cdot c_i') f^z = \sum_{i=1}^n (c_i\varepsilon^{z_i}\otimes c_i') f^z\\
&= \sum_{i=1}^n (c_i\otimes c_i')(\varepsilon\otimes \mathrm{id})^z f^z = c\Delta^z(\varepsilon\otimes\mathrm{id})^z f^z = cf^z\\
\sum_{i=1}^n c_i f^{z_i}\cdot c_i'\varepsilon^{z_i'} &= \sum_{i=1}^n (c_i\cdot c_i'\varepsilon^{z_i'}) f^z = \sum_{i=1}^n (c_i\otimes c_i'\varepsilon^{z_i'}) f^z\\
&= \sum_{i=1}^n (c_i\otimes c_i')(\mathrm{id}\otimes\varepsilon)^z f^z = c\Delta^z(\mathrm{id}\otimes\varepsilon)^z f^z = cf^z
\end{aligned}
$$

and finally since f is a coalgebra morphism

$$
\begin{aligned}
\sum_{i=1}^n (0, c_i f^{z_i})\otimes(0, c_i' f^{z_i'}) &= \sum_{i=1}^n (c_i\otimes c_i')(f\otimes f)^z(\iota\otimes\iota)^z\\
&= c\Delta^z(f\otimes f)^z(\iota\otimes\iota)^z = cf^z\Delta^z(\iota\otimes\iota)^z.
\end{aligned}
$$

Therefore $\Delta(\bar f\otimes\bar f) = \bar f\hat\Delta$, i.e. $\bar f$ is a coalgebra morphism.
Moreover, since $c\bar f^z\hat\varepsilon^z = (c\varepsilon^z, cf^z)\hat\varepsilon^z = c\varepsilon^z$, we have $\bar f\hat\varepsilon = \varepsilon$. It follows that $\bar f$ is a morphism of counital coalgebras.

(3) The statements of (1) and (2) together are equivalent to the assertion V is left adjoint to E, cf. Lemma 2. □

1.3.5 Counital tensor coalgebras

Remark 32 Let A be a graded module. For the tensor coalgebra $TA = \bigoplus_{k\geq 1} A^{\otimes k}$ attaching a counit yields the counital tensor coalgebra $\hat{T}A := E(TA) = \dot{R} \oplus TA = \bigoplus_{k\geq 0} A^{\otimes k}$. We write $\iota_k\colon A^{\otimes k} \to \hat{T}A$ and $\pi_k\colon \hat{T}A \to A^{\otimes k}$ for the inclusion and projection of the k-th direct summand, where $k \geq 0$.
For $k, \ell_1, \ell_2 \geq 0$ the following hold.

(1) $\iota_k \hat{\Delta}(\pi_{\ell_1} \otimes \pi_{\ell_2}) = \left\{\begin{matrix} \mathrm{id}_A^{\otimes k} & \text{if } k = \ell_1 + \ell_2 \\ 0 & \text{else} \end{matrix}\right\} : A^{\otimes k} \to A^{\otimes \ell_1} \otimes A^{\otimes \ell_2} = A^{\otimes(\ell_1+\ell_2)}$

(2) $\hat{\Delta}(\pi_{\ell_1} \otimes \pi_{\ell_2}) = \pi_{\ell_1+\ell_2}$

(3) $\iota_k \hat{\Delta} = \sum_{\substack{i+j=k \\ i,j\geq 0}} \iota_i \otimes \iota_j$

(4) Given a morphism of coalgebras $f\colon TA \to TB$ between the tensor coalgebras over the graded modules A and B, the morphism $\hat{f} = Ef\colon \hat{T}A \to \hat{T}B$ between the counital tensor coalgebras satisfies for $k, \ell \geq 0$

$$\hat{f}_{k,\ell} = \iota_k \hat{f} \pi_\ell = \left\{\begin{matrix} f_{k,\ell} & \text{if } k, \ell \geq 1 \\ \mathrm{id}_{\dot{R}} & \text{if } k = \ell = 0 \\ 0 & \text{else} \end{matrix}\right\} : A^{\otimes k} \to B^{\otimes \ell}.$$

Proof. (1) By definition of the comultiplication on $\hat{T}A = E(TA)$ we have for an element $(r, a) \in (\dot{R} \oplus TA)^z = (\hat{T}A)^z$ for $z \in \mathrm{Mor}(\mathcal{Z})$ that

$$(r, a)\hat{\Delta} = (r, 0) \otimes (1, 0) + (1, 0) \otimes (0, a) + (0, a) \otimes (1, 0) + a\Delta^z(\iota \otimes \iota)^z$$

where $\iota\colon TA \to \dot{R} \oplus TA$ is the inclusion into the second summand. Hence if $k = 0$ we obtain for $r \in (\dot{R})^z$ for $z \in \mathrm{Mor}(\mathcal{Z})$ and $\ell_1, \ell_2 \geq 0$

$$r\iota_0^z \hat{\Delta}^z(\pi_{\ell_1} \otimes \pi_{\ell_2})^z = (r, 0)\hat{\Delta}^z(\pi_{\ell_1} \otimes \pi_{\ell_2})^z = \big((r, 0) \otimes (1, 0)\big)(\pi_{\ell_1} \otimes \pi_{\ell_2})^z = \begin{cases} r & \text{for } \ell_1, \ell_2 = 0 \\ 0 & \text{else.} \end{cases}$$

If $k \geq 1$ we have for $a \in \left(A^{\otimes k}\right)^z$ for $z \in \mathrm{Mor}(\mathcal{Z})$ and $\ell_1, \ell_2 \geq 0$

$$\begin{aligned} a\iota_k^z \hat{\Delta}^z(\pi_{\ell_1} \otimes \pi_{\ell_2})^z &= (0, a)\hat{\Delta}^z(\pi_{\ell_1} \otimes \pi_{\ell_2})^z \\ &= \big((1, 0) \otimes (0, a\iota_k^z) + (0, a\iota_k^z) \otimes (1, 0) + a\iota_k^z \Delta^z(\iota \otimes \iota)^z\big)(\pi_{\ell_1} \otimes \pi_{\ell_2})^z. \end{aligned}$$

If $\ell_1 = 0$ or $\ell_2 = 0$, then $\iota\pi_{\ell_1} = 0$ or $\iota\pi_{\ell_2} = 0$. So the above expression is only non-zero if either $\ell_1 = 0$ and $\ell_2 = k$ or $\ell_1 = k$ and $\ell_2 = 0$, in both cases it equals $a\iota_k$.
If $\ell_1 \geq 1$ and $\ell_2 \geq 1$, the above expression equals $a\iota_k^z \Delta^z(\pi_{\ell_1} \otimes \pi_{\ell_2})$ and the assertion follows from Remark 20.
The assertions of (2) and (3) now follow from (1).
(4) By definition, we have for $(r, a) \in (\dot{R} \oplus TA)^z = (\hat{T}A)^z$ for $z \in \mathrm{Mor}(\mathcal{Z})$ that $(r, a)\hat{f} = (r, af)$. Since $r\iota_0 = (r, 0)$ and $a\iota_k = (0, a\iota_k)$ for $k \geq 1$ the assertion follows. □

Lemma 33 *Let A and B be graded modules and suppose given a morphism of coalgebras $f\colon TA \to TB$. Then for $k, \ell_1, \ell_2 \geq 0$ we have*

$$\hat{f}_{k,\ell_1+\ell_2} = \sum_{\substack{i+j=k \\ i,j\geq 0}} \hat{f}_{i,\ell_1} \otimes \hat{f}_{j,\ell_2} \, : A^{\otimes k} \to B^{\otimes \ell_1} \otimes B^{\otimes \ell_2} = B^{\otimes(\ell_1+\ell_2)}$$

Proof. We use the description of $\hat{\Delta}$ on the counital tensor coalgebra from Remark 32. For the left-hand side, consider

$$\hat{f}_{k,\ell_1+\ell_2} = \iota_k \hat{f} \pi_{\ell_1+\ell_2} = \iota_k \hat{f} \hat{\Delta}(\pi_{\ell_1} \otimes \pi_{\ell_2}).$$

For the right-hand side, consider

$$\sum_{\substack{i+j=k \\ i,j\geq 0}} \hat{f}_{i,\ell_1} \otimes \hat{f}_{j,\ell_2} = \sum_{i=0}^{k} (\iota_i \otimes \iota_{k-i})(\hat{f} \otimes \hat{f})(\pi_{\ell_1} \otimes \pi_{\ell_2}) = \iota_k \hat{\Delta}(\hat{f} \otimes \hat{f})(\pi_{\ell_1} \otimes \pi_{\ell_2}).$$

Since $\hat{f}$ is a morphism of coalgebras, the assertion follows. □

Chapter 2

A_∞-homotopies

Throughout this chapter, let R be a commutative ring.

All modules are left R-modules, all linear maps between modules are R-linear maps, all tensor products of modules are tensor products over R.

Fix a grading category $\mathcal{Z}$. Unless stated otherwise, by *graded* we mean $\mathcal{Z}$-*graded*.

2.1 Coderivations

In the previous sections §1.2 and §1.3 we showed how one constructs the category A_∞-alg of A_∞-algebras and morphisms of A_∞-algebras together with a full and faithful functor

$$\mathrm{Bar}\colon \mathrm{A}_\infty\text{-}\mathsf{alg} \to \mathsf{dgCoalg}$$

into the category $\mathsf{dgCoalg}$ of differential graded coalgebras, cf. Definition 28.

Via this functor, the category A_∞-alg is equivalent to the full subcategory $\mathsf{dtCoalg}$ of $\mathsf{dgCoalg}$ of differential graded tensor coalgebras, cf. Definition 29.

We want to arrive at a definition of homotopies between A_∞-morphisms. Using the equivalence of A_∞-alg and $\mathsf{dtCoalg}$ described above, it suffices to define homotopies of differential graded coalgebra morphisms between tensor coalgebras.

In analogy to the usual homotopy of complex morphisms, we shall define a homotopy between differential graded coalgebra morphisms $f\colon TA \to TB$ and $g\colon TA \to TB$ to be a graded linear map $h\colon TA \to TB$ of degree -1 that satisfies $f - g = hm + mh$ and that is in some sense compatible with the comultiplications on TA and TB.

We will generalise the notion of a coderivation to the notion of an (f, g)-coderivation. The requirement on h to be such an (f, g)-coderivation will be the additional compatibility condition.

In this section we present basic properties of these generalised coderivations between tensor coalgebras and show how they assemble into an A_∞-category.

2.1.1 Definition and first properties

Suppose given graded coalgebras (C, Δ) and (D, Δ).

Definition 34 Let $f\colon C \to D$ and $g\colon C \to D$ be morphisms of graded coalgebras. A graded linear map $h\colon C \to D$ of degree $p \in \mathbf{Z}$ is an (f,g)*-coderivation of degree* p if it satisfies

$$h\Delta = \Delta(f \otimes h + h \otimes g).$$

We denote by $\mathrm{Coder}(C,D)^{p,(f,g)}$ the module of (f,g)-coderivations of degree p.

Remark 35 Let $f\colon C \to D$ and $g\colon C \to D$ be morphisms of graded coalgebras.
Then the graded linear map $h_{f,g} := f - g$ is an (f,g)-coderivation of degree 0.

Proof. We have

$$\begin{aligned} h_{f,g}\Delta = (f-g)\Delta &= \Delta(f \otimes f - g \otimes g) \\ &= \Delta(f \otimes (f-g) + (f-g) \otimes g) = \Delta(f \otimes h_{f,g} + h_{f,g} \otimes g). \end{aligned}$$ □

Lemma 36 *Suppose given graded coalgebras B, C, D and E with morphisms of coalgebras between them as in the following diagram.*

$$B \xrightarrow{\ s\ } C \overset{f}{\underset{g}{\rightrightarrows}} D \xrightarrow{\ t\ } E$$

Suppose given an (f,g)-coderivation $h\colon C \to D$ of degree $p \in \mathbf{Z}$. Then $sht\colon B \to E$ is an (sft, sgt)-coderivation of degree p.

Proof. As morphisms of graded coalgebras have degree 0, the graded linear map sht has degree p. It remains to verify that sht is an (sft, sgt)-coderivation. We calculate.

$$\begin{aligned} sht\Delta = sh\Delta(t \otimes t) &= s\Delta(f \otimes h + h \otimes g)(t \otimes t) \\ &= \Delta(s \otimes s)(f \otimes h + h \otimes g)(t \otimes t) = \Delta(sft \otimes sht + sht \otimes sgt) \end{aligned}$$

It follows that sht is an (sft, sgt)-coderivation of degree p. □

Lemma 37 (Lifting to coderivations) *Let A and B be graded modules.*
Let $f\colon TA \to TB$ and $g\colon TA \to TB$ be morphisms of graded coalgebras between the tensor coalgebras over A and B. Let $p \in \mathbf{Z}$.
Consider the linear map

$$\begin{aligned} \beta\colon \quad \mathrm{Coder}(TA,TB)^{p,(f,g)} &\longrightarrow \mathsf{grHom}(TA,B)^p \\ h &\longmapsto h\pi_1. \end{aligned}$$

from the module of (f,g)-coderivations from TA to TB of degree p to the module of graded linear maps from TA to B of degree p.
Recall that for a coalgebra morphism $f\colon TA \to TB$ we write $\hat{f} = Ef\colon \hat{T}A \to \hat{T}B$ for the corresponding morphism between the counital tensor coalgebras, cf. Remark 32.
Consider the map $\alpha\colon \mathsf{grHom}(TA,B)^p \to \mathrm{Coder}(TA,TB)^{p,(f,g)}$ that is for a graded linear map $\eta\colon TA \to B$ of degree p given by

$$(\eta\alpha)_{k,\ell} = \sum_{\substack{r+s+t=k \\ r'+1+t'=\ell \\ r,r',t,t' \geq 0,\, s \geq 1}} \hat{f}_{r,r'} \otimes \eta_s \otimes \hat{g}_{t,t'} \ : A^{\otimes k} \to B^{\otimes \ell},$$

where $k, \ell \geq 1$.

Then α and β are mutually inverse linear isomorphisms.

In particular, for an (f, g)-coderivation $h\colon TA \to TB$ of degree p the following formula holds for $k, \ell \geq 1$.

$$h_{k,\ell} = \sum_{\substack{r+s+t=k \\ r'+1+t'=\ell \\ r,r',t,t'\geq 0,\, s\geq 1}} \hat{f}_{r,r'} \otimes h_{s,1} \otimes \hat{g}_{t,t'} = \sum_{\substack{r+s+t=k \\ r'+1+t'=\ell \\ r\geq r'\geq 0,\, t\geq t'\geq 0,\, s\geq 1}} \hat{f}_{r,r'} \otimes h_{s,1} \otimes \hat{g}_{t,t'}$$

Moreover, $h_{k,\ell} = 0$ if $k < \ell$.

Proof. We show that α is well-defined. Let $\eta\colon TA \to B$ be a graded linear map of degree p. To show that $\eta\alpha$ is well-defined as a graded linear map, we have to show that for $k \geq 1$ there only finitely many $\ell \geq 1$ such that $(\eta\alpha)_{k,\ell} \neq 0$.

We *claim* that $(\eta\alpha)_{k,\ell} = 0$ for $\ell > k$. Indeed, given $r, r', t, t' \geq 0$ and $s \geq 1$ with $r + s + t = k$ and $r' + 1 + t' = \ell$ this means that either $r' > r$ or $t' > t$. By Lemma 23 a coalgebra morphism f satisfies $f_{i,j} = 0$ whenever $j > i$ and using Remark 32 also $\hat{f}$ satisfies $\hat{f}_{i,j} = 0$ whenever $j > i$. Hence for $\ell > k$ we have

$$(\eta\alpha)_{k,\ell} = \sum_{\substack{r+s+t=k \\ r'+1+t'=\ell \\ r,r',t,t'\geq 0,\, s\geq 1}} \hat{f}_{r,r'} \otimes \eta_s \otimes \hat{g}_{t,t'} = 0.$$

This shows the *claim*. In particular, $\eta\alpha\colon TA \to TB$ is a well-defined graded linear map.

It remains to show that $\eta\alpha$ is an (f, g)-coderivation, i.e. it remains to show that $\eta\alpha$ satisfies $(\eta\alpha)\Delta = \Delta(f \otimes (\eta\alpha) + (\eta\alpha) \otimes g)$. It suffices to show that

$$\iota_k(\eta\alpha)\Delta(\pi_{\ell_1} \otimes \pi_{\ell_2}) = \iota_k\Delta(f \otimes (\eta\alpha) + (\eta\alpha) \otimes g)(\pi_{\ell_1} \otimes \pi_{\ell_2})$$

for $k, \ell_1, \ell_2 \geq 1$. Using Remark 20 we obtain for the left-hand side

$$\begin{aligned} \iota_k(\eta\alpha)\Delta(\pi_{\ell_1} \otimes \pi_{\ell_2}) &= \iota_k(\eta\alpha)\pi_{\ell_1+\ell_2} \\ &= (\eta\alpha)_{k,\ell_1+\ell_2} \end{aligned}$$

and similarly for the right-hand side

$$\begin{aligned} \iota_k\Delta(f \otimes (\eta\alpha) + (\eta\alpha) \otimes g)(\pi_{\ell_1} \otimes \pi_{\ell_2}) &= \sum_{\substack{i+j=k \\ i,j\geq 1}} (\iota_i \otimes \iota_j)(f \otimes (\eta\alpha) + (\eta\alpha) \otimes g)(\pi_{\ell_1} \otimes \pi_{\ell_2}) \\ &= \sum_{\substack{i+j=k \\ i,j\geq 1}} (\iota_i \otimes \iota_j)(f \otimes (\eta\alpha))(\pi_{\ell_1} \otimes \pi_{\ell_2}) \\ &\quad + \sum_{\substack{i+j=k \\ i,j\geq 1}} (\iota_i \otimes \iota_j)((\eta\alpha) \otimes g)(\pi_{\ell_1} \otimes \pi_{\ell_2}) \\ &= \sum_{\substack{i+j=k \\ i,j\geq 1}} f_{i,\ell_1} \otimes (\eta\alpha)_{j,\ell_2} + \sum_{\substack{i+j=k \\ i,j\geq 1}} (\eta\alpha)_{i,\ell_1} \otimes g_{j,\ell_2}. \end{aligned}$$

Using Remark 32 and Lemma 33 we obtain

$$
\begin{aligned}
&(\eta\alpha)_{k,\ell_1+\ell_2}\\
&= \sum_{\substack{r+s+t=k\\ r'+1+t'=\ell_1+\ell_2\\ r,r',t,t'\geq 0,\, s\geq 1}} \hat{f}_{r,r'}\otimes \eta_s \otimes \hat{g}_{t,t'}\\
&= \sum_{\substack{r+s+t=k\\ r'+1+t'=\ell_1+\ell_2\\ r'\geq \ell_1,\, r,t,t'\geq 0,\, s\geq 1}} \hat{f}_{r,r'}\otimes \eta_s \otimes \hat{g}_{t,t'} + \sum_{\substack{r+s+t=k\\ r'+1+t'=\ell_1+\ell_2\\ r,r',t\geq 0,\, t'\geq \ell_2,\, s\geq 1}} \hat{f}_{r,r'}\otimes \eta_s \otimes \hat{g}_{t,t'}\\
&= \sum_{\substack{r+s+t=k\\ u'+1+t'=\ell_2\\ r,u',t,t'\geq 0,\, s\geq 1}} \hat{f}_{r,\ell_1+u'}\otimes \eta_s \otimes \hat{g}_{t,t'} + \sum_{\substack{r+s+t=k\\ r'+1+v'=\ell_1\\ r,r',t,v'\geq 0,\, s\geq 1}} \hat{f}_{r,r'}\otimes \eta_s \otimes \hat{g}_{t,v'+\ell_2}\\
&= \sum_{\substack{r+s+t=k\\ u'+1+t'=\ell_2\\ r,u',t,t'\geq 0,\, s\geq 1}} \sum_{\substack{i+i'=r\\ i,i'\geq 0}} \hat{f}_{i,\ell_1}\otimes \hat{f}_{i',u'}\otimes \eta_s \otimes \hat{g}_{t,t'} + \sum_{\substack{r+s+t=k\\ r'+1+v'=\ell_1\\ r,r',t,v'\geq 0,\, s\geq 1}} \sum_{\substack{j'+j=t\\ j',j\geq 0}} \hat{f}_{r,r'}\otimes \eta_s \otimes \hat{g}_{j',v'}\otimes \hat{g}_{j,\ell_2}\\
&= \sum_{\substack{i+i'+s+t=k\\ u'+1+t'=\ell_2\\ i,i',u',t,t'\geq 0,\, s\geq 1}} \hat{f}_{i,\ell_1}\otimes \hat{f}_{i',u'}\otimes \eta_s \otimes \hat{g}_{t,t'} + \sum_{\substack{r+s+j'+j=k\\ r'+1+v'=\ell_1\\ r,r',j',j,v'\geq 0,\, s\geq 1}} \hat{f}_{r,r'}\otimes \eta_s \otimes \hat{g}_{j',v'}\otimes \hat{g}_{j,\ell_2}\\
&= \sum_{\substack{i+i'+s+t=k\\ u'+1+t'=\ell_2\\ i\geq 1,\, i',u',t,t'\geq 0,\, s\geq 1}} f_{i,\ell_1}\otimes \hat{f}_{i',u'}\otimes \eta_s \otimes \hat{g}_{t,t'} + \sum_{\substack{r+s+j'+j=k\\ r'+1+v'=\ell_1\\ j\geq 1,\, r,r',j',v'\geq 0,\, s\geq 1}} \hat{f}_{r,r'}\otimes \eta_s \otimes \hat{g}_{j',v'}\otimes g_{j,\ell_2}\\
&= \sum_{\substack{i+j=k\\ i,j\geq 1}} \sum_{\substack{i'+s+t=j\\ u'+1+t'=\ell_2\\ i',u',t,t'\geq 0,\, s\geq 1}} f_{i,\ell_1}\otimes \hat{f}_{i',u'}\otimes \eta_s \otimes \hat{g}_{t,t'} + \sum_{\substack{i+j=k\\ i,j\geq 1}} \sum_{\substack{r+s+j'=i\\ r'+1+v'=\ell_1\\ r,r',j',v'\geq 0,\, s\geq 1}} \hat{f}_{r,r'}\otimes \eta_s \otimes \hat{g}_{j',v'}\otimes g_{j,\ell_2}\\
&= \sum_{\substack{i+j=k\\ i,j\geq 1}} f_{i,\ell_1}\otimes (\eta\alpha)_{j,\ell_2} + \sum_{\substack{i+j=k\\ i,j\geq 1}} (\eta\alpha)_{i,\ell_1}\otimes g_{j,\ell_2}.
\end{aligned}
$$

Hence $\eta\alpha$ is an (f,g)-coderivation, i.e. α is well-defined.

We show that $\alpha\beta = \mathrm{id}$. For this, let $\eta\colon TA \to B$ be a graded linear map of degree p. We have to show that $(\eta\alpha)\beta = (\eta\alpha)\pi_1 = \eta$. It suffices to verify that for $k \geq 1$ the equation $\iota_k(\eta\alpha)\pi_1 = (\eta\alpha)_{k,1} = \eta_k = \iota_k\eta$ holds. By definition of α we have using Remark 32

$$
(\eta\alpha)_{k,1} = \sum_{\substack{r+s+t=k\\ r'+1+t'=1\\ r,r',t,t'\geq 0,\, s\geq 1}} \hat{f}_{r,r'}\otimes \eta_s \otimes \hat{g}_{t,t'} = \sum_{\substack{r+s+t=k\\ r,t\geq 0,\, s\geq 1}} \hat{f}_{r,0}\otimes \eta_s \otimes \hat{g}_{t,0} = \hat{f}_{0,0}\otimes \eta_k \otimes \hat{g}_{0,0} = \eta_k.
$$

We show that β is injective. For this, we show that its kernel is trivial. Let $h\colon TA \to TB$ be an (f,g)-coderivation of degree p such that $h\beta = h\pi_1 = 0$. We have to show that $h = 0$. It suffices to verify that $\iota_k h = 0$ holds for $k \geq 1$. We proceed by induction on k.

For $k = 1$ we have $\iota_1 h\Delta = \iota_1\Delta(f\otimes h + h\otimes g) = 0$, since h is an (f,g)-coderivation and $\iota_1\Delta = 0$. Using Remark 19 we conclude that $\iota_1 h = \iota_1 h\pi_1\iota_1 = \iota_1(h\beta)\iota_1 = 0$.

Now let $k > 1$ and assume that $\iota_\ell h = 0$ for $\ell < k$. Since h is an (f,g)-coderivation we have using Remark 20

$$\begin{aligned}\iota_k h\Delta = \iota_k\Delta(f\otimes h + h\otimes g) &= \sum_{\substack{i+j=k\\ i,j\geq 1}} (\iota_i\otimes\iota_j)(f\otimes h + h\otimes g)\\ &= \sum_{\substack{i+j=k\\ i,j\geq 1}} (\iota_i f\otimes \iota_j h + \iota_i h\otimes \iota_j g) = 0\end{aligned}$$

In the sum on the right hand side, both i and j are strictly smaller than k, hence all summands are zero by induction. It follows that $\iota_k h\Delta = 0$, so again using Remark 19 we conclude that $\iota_k h = \iota_k h\pi_1\iota_1 = \iota_k(h\beta)\iota_1 = 0$.

Hence β is an injective linear map with $\alpha\beta = \mathrm{id}$. Therefore α and β are mutually inverse linear isomorphisms.

For an (f,g)-coderivation $h\colon TA\to TB$ of degree p we have

$$h_{k,\ell} = (h\beta\alpha)_{k,\ell} = \sum_{\substack{r+s+t=k\\ r'+1+t'=\ell\\ r,r',t,t'\geq 0,\, s\geq 1}} \hat{f}_{r,r'}\otimes(h\beta)_s\otimes\hat{g}_{t,t'} = \sum_{\substack{r+s+t=k\\ r'+1+t'=\ell\\ r,r',t,t'\geq 0,\, s\geq 1}} \hat{f}_{r,r'}\otimes h_{s,1}\otimes\hat{g}_{t,t'}$$

for $k,\ell\geq 1$. Here we used that $(h\beta)_i = (h\pi_1)_i = \iota_i h\pi_1 = h_{i,1}$.

Finally, at the beginning of this proof we showed that for a graded linear map $\eta\colon TA\to B$ of degree p one has $(\eta\alpha)_{k,\ell} = 0$ whenever $\ell > k$. Since $h_{k,\ell} = (h\beta\alpha)_{k,\ell}$, it follows that also $h_{k,\ell} = 0$ whenever $\ell > k$. □

Corollary 38 *In the situation of the previous Lemma 37, let $h\colon TA\to TB$ and $\tilde{h}\colon TA\to TB$ be (f,g)-coderivations of degree p and let $k,\ell\geq 1$.*
Suppose that $h_{s,1} = \tilde{h}_{s,1}$ for $1\leq s\leq k-\ell+1$. Then $h_{k,\ell} = \tilde{h}_{k,\ell}$.

Proof. This follows from the second formula for $h_{k,\ell}$ in Lemma 37. □

Corollary 39 *In the situation of Lemma 37, the inclusion*

$$j\colon\quad \mathrm{Coder}(TA,TB)^{p,(f,g)} \hookrightarrow \mathsf{grHom}(TA,TB)^p$$

is a split monomorphism.

Proof. Using the α from Lemma 37, we define the linear map

$$\begin{aligned} r\colon\quad \mathsf{grHom}(TA,TB)^p &\longrightarrow \mathrm{Coder}(TA,TB)^{p,(f,g)}\\ \varphi &\longmapsto (\varphi\pi_1)\alpha. \end{aligned}$$

For an (f,g)-coderivation $h\colon TA\to TB$ of degree p we have again using Lemma 37

$$hjr = ((hj)\pi_1)\alpha = h\beta\alpha = h.$$

Hence $jr = \mathrm{id}$, i.e. j is a split monomorphism. □

2.1.2 The complex of coderivations

Let (C, Δ, m) and (D, Δ, m) be differential graded coalgebras.

Lemma 40

(1) *The* $\mathbf{Z}$*-graded linear map*

$$\begin{array}{rrcl} \mu\colon & \mathsf{grHom}(C,D) & \longrightarrow & \mathsf{grHom}(C,D) \\ \mu^p\colon & \varphi & \longmapsto & \varphi m - (-1)^p m\varphi \end{array}$$

is a differential on $\mathsf{grHom}(C, D)$*, i.e. it is of degree* 1 *and satisfies* $\mu^2 = 0$*.*

(2) *Suppose given a graded linear map* $\varphi\colon C \to D$ *of degree* $p \in \mathbf{Z}$*. Suppose given* $k \geq 1$ *and graded linear maps* $\varphi_i\colon C \to D$ *of degree* $p_i \in \mathbf{Z}$ *and* $\varphi'_i\colon C \to D$ *of degree* $p'_i \in \mathbf{Z}$ *for* $1 \leq i \leq k$ *such that* $\varphi\Delta = \sum_{i=1}^k \Delta(\varphi_i \otimes \varphi'_i)$*. In particular, we have* $p_i + p'_i = p$ *for* $1 \leq i \leq k$*.*
Then the following equation holds.

$$(\varphi\mu^p)\Delta = \sum_{i=1}^k \Delta\big(\varphi_i \otimes (\varphi'_i\mu^{p'_i}) + (-1)^{p'_i}(\varphi_i\mu^{p_i}) \otimes \varphi'_i\big)$$

Proof. (1) For a graded linear map $\varphi\colon C \to D$ of degree p, the map $\varphi m - (-1)^p m\varphi$ is a graded linear map of degree $p+1$. It remains to verify the differential condition $\mu^2 = 0$.

$$\begin{aligned} \varphi\mu^2 &= (\varphi m - (-1)^p m\varphi)\mu \\ &= (\varphi m)\mu - (-1)^p (m\varphi)\mu \\ &= \varphi mm - (-1)^{p+1} m\varphi m - (-1)^p (m\varphi m - (-1)^{p+1} mm\varphi) \\ &= (-1)^p m\varphi m - (-1)^p m\varphi m \\ &= 0. \end{aligned}$$

(2) Recall that m is an $(\mathrm{id}, \mathrm{id})$-coderivation, i.e. it satisfies $m\Delta = \Delta(\mathrm{id} \otimes m + m \otimes \mathrm{id})$. Note that we have to take the Koszul sign rule into consideration. We calculate.

$$\begin{aligned} (\varphi\mu)\Delta &= (\varphi m - (-1)^p m\varphi)\Delta \\ &= \varphi\Delta(\mathrm{id} \otimes m + m \otimes \mathrm{id}) - \sum_{i=1}^k (-1)^p m\Delta(\varphi_i \otimes \varphi'_i) \\ &= \sum_{i=1}^k \Delta(\varphi_i \otimes \varphi'_i)(\mathrm{id} \otimes m + m \otimes \mathrm{id}) - \sum_{i=1}^k (-1)^p \Delta(\mathrm{id} \otimes m + m \otimes \mathrm{id})(\varphi_i \otimes \varphi'_i) \\ &= \sum_{i=1}^k \Delta\big(\varphi_i \otimes \varphi'_i m + (-1)^{p'_i}(\varphi_i m \otimes \varphi'_i)\big) - \sum_{i=1}^k (-1)^p \Delta((-1)^{p_i}(\varphi_i \otimes m\varphi'_i) + m\varphi_i \otimes \varphi'_i) \\ &= \sum_{i=1}^k \Delta\big(\varphi_i \otimes \varphi'_i m - (-1)^{p+p_i}(\varphi_i \otimes m\varphi'_i) + (-1)^{p'_i}(\varphi_i m \otimes \varphi'_i) - (-1)^p(m\varphi_i \otimes \varphi'_i)\big) \\ &= \sum_{i=1}^k \Delta\big(\varphi_i \otimes (\varphi'_i m - (-1)^{p'_i} m\varphi'_i) + (-1)^{p'_i}(\varphi_i m - (-1)^{p_i} m\varphi_i) \otimes \varphi'_i\big) \\ &= \sum_{i=1}^k \Delta\big(\varphi_i \otimes (\varphi'_i\mu) + (-1)^{p'_i}(\varphi_i\mu) \otimes \varphi'_i\big) \end{aligned}$$

□

Definition 41

(1) We define the grading category $\mathcal{Z}_{C,D} := \mathbf{Z} \times \mathsf{Pair}(\mathsf{dgCoalg}(C,D))$, cf. Example 4 and Definition 5.

(2) We define the $\mathcal{Z}_{C,D}$-graded module of *precoderivations* $\operatorname{PreCoder}(C,D)$ that has at $(p,(f,g))$ the module

$$\operatorname{PreCoder}(C,D)^{p,(f,g)} := \mathsf{grHom}(C,D)^p = \{\varphi\colon C \to D : f \text{ is a graded linear map of degree } p\}$$

for $p \in \mathbf{Z}$ and differential graded coalgebra morphisms $f, g \in \mathsf{dgCoalg}(C,D)$.

(3) We define the $\mathcal{Z}_{C,D}$-graded module of *coderivations* $\operatorname{Coder}(C,D)$ that has at $(p,(f,g))$ the module of (f,g)-coderivations of degree p, i.e.

$$\operatorname{Coder}(C,D)^{p,(f,g)} := \left\{ h\colon C \to D : \begin{array}{r} h \text{ is a graded linear map of degree } p \\ \text{and satisfies } h\Delta = \Delta(f \otimes h + h \otimes g) \end{array} \right\}$$

for $p \in \mathbf{Z}$ and differential graded coalgebra morphisms $f, g \in \mathsf{dgCoalg}(C,D)$.
Note that $\operatorname{Coder}(C,D) \subseteq \operatorname{PreCoder}(C,D)$.

Lemma 42 *Consider the $\mathcal{Z}_{C,D}$-graded coderivation*

$$\mathfrak{m}\colon \quad T\operatorname{PreCoder}(C,D) \longrightarrow T\operatorname{PreCoder}(C,D)$$

on the tensor coalgebra $(T\operatorname{PreCoder}(C,D),\Delta)$ over $\operatorname{PreCoder}(C,D)$ with $\mathfrak{m}_{1,1}^{p,(f,g)} = \mu^p$ and with $\mathfrak{m}_{k,1}^{p,(f,g)} = 0$ for $k \geq 2$, where $p \in \mathbf{Z}$ and $f, g \in \mathsf{dgCoalg}(C,D)$, cf. Lemma 22.(2).
Then $(T\operatorname{PreCoder}(C,D),\Delta,\mathfrak{m})$ is a differential $\mathcal{Z}_{C,D}$-graded coalgebra.

Proof. It remains to show that $\mathfrak{m}$ is a differential, i.e. that $\mathfrak{m}^2 = 0$. By Lemma 24.(1) this is equivalent to

$$0 = \sum_{\substack{r+s+t=k \\ r,t \geq 0,\, s \geq 1}} (\mathrm{id}^{\otimes r} \otimes \mathfrak{m}_{s,1} \otimes \mathrm{id}^{\otimes t})\mathfrak{m}_{r+1+t,1}$$

for $k \geq 1$. But since $\mathfrak{m}_{k,1} = 0$ for $k \geq 2$, this condition reduces to $\mathfrak{m}_{1,1}\mathfrak{m}_{1,1} = 0$. However, by Lemma 40.(1) the graded linear map μ is a differential, i.e. it satisfies $\mu^p\mu^{p+1} = 0$ for $p \in \mathbf{Z}$. Since $\mathfrak{m}_{1,1}^{p,(f,g)} = \mu^p$, also $\mathfrak{m}_{1,1}$ is a differential, i.e. satisfies $\mathfrak{m}_{1,1}^{p,(f,g)}\mathfrak{m}_{1,1}^{p+1,(f,g)} = 0$ for $p \in \mathbf{Z}$ and morphisms of differential graded coalgebras $f, g \in \mathsf{dgCoalg}(C,D)$. □

2.1.3 Tensoring coderivations

Let A and B be graded modules.
Recall the tensor coalgebras (TA,Δ) and (TB,Δ) over A and B, cf. Definition 18.

Definition 43 Let $n \geq 1$. Suppose given morphisms of graded coalgebras $f_i\colon TA \to TB$ for $0 \leq i \leq n$. Suppose given $p_i \in \mathbf{Z}$ for $1 \leq i \leq n$ and let $p := \sum_{i=1}^n p_i$. Define the linear map

$$\tau_n\colon \quad \operatorname{Coder}(TA,TB)^{p_1,(f_0,f_1)} \otimes \ldots \otimes \operatorname{Coder}(TA,TB)^{p_n,(f_{n-1},f_n)} \longrightarrow \mathsf{grHom}(TA,TB)^p$$

for $h_i \in \mathrm{Coder}(TA,TB)^{p_i,(f_{i-1},f_i)}$ for $1 \le i \le n$ by

$$\begin{aligned}
\big((h_1 \otimes \ldots \otimes h_n)\tau_n\big)_{k,\ell} &= \sum_{\substack{r_0 + \left(\sum_{\beta=1}^{n} s_\beta + r_\beta\right) = k \\ r'_0 + \left(\sum_{\beta=1}^{n} 1 + r'_\beta\right) = \ell \\ r_0,\ldots,r_n,r'_0,\ldots,r'_n \ge 0,\, s_1,\ldots,s_n \ge 1}} (\hat{f}_0)_{r_0,r'_0} \otimes \bigotimes_{\beta=1}^{n} \Big((h_\beta)_{s_\beta,1} \otimes (\hat{f}_\beta)_{r_\beta,r'_\beta} \Big) \\
&= \sum_{\substack{\left(\sum_{\beta=1}^{n} r_{\beta-1} + s_\beta\right) + r_n = k \\ \left(\sum_{\beta=1}^{n} r'_{\beta-1} + 1\right) + r'_n = \ell \\ r_0,\ldots,r_n,r'_0,\ldots,r'_n \ge 0,\, s_1,\ldots,s_n \ge 1}} \bigotimes_{\beta=1}^{n} \Big((\hat{f}_{\beta-1})_{r_{\beta-1},r'_{\beta-1}} \otimes (h_\beta)_{s_\beta,1} \Big) \otimes (\hat{f}_n)_{r_n,r'_n}
\end{aligned}$$

for $k, \ell \ge 1$.

Note that by Remark 44 below, given $k \ge 1$ there are only finitely many $\ell \ge 1$ such that $\big((h_1 \otimes \ldots \otimes h_n)\tau_n\big)_{k,\ell} \ne 0$. Hence $(h_1 \otimes \ldots \otimes h_n)\tau_n$ is well-defined as a graded linear map.

Remark 44 Suppose given the situation as in Definition 43.

(1) If $k < \ell$, one has $\big((h_1 \otimes \ldots \otimes h_n)\tau_n\big)_{k,\ell} = \iota_k\big((h_1 \otimes \ldots \otimes h_n)\tau_n\big)\pi_\ell = 0$.

(2) If $\ell < n$, one has $\big((h_1 \otimes \ldots \otimes h_n)\tau_n\big)\pi_\ell = 0$.

Proof. (1) Using Lemma 23.(1) and Remark 32.(4) it follows that one has $(\hat{f}_i)_{k,\ell} = 0$ whenever $k < \ell$. So a summand in the formula for $\big((h_1 \otimes \ldots h_n)\tau_n\big)_{k,\ell}$ in Definition 43 is non-zero only if $r_\beta \ge r'_\beta$ for $1 \le \beta \le n$, which implies that $k = r_0 + \big(\sum_{\beta=1}^{n} r_\beta + s_\beta\big) \ge r'_0 + \big(\sum_{\beta=1}^{n} 1 + r'_\beta\big) = \ell$. Therefore we have $\big((h_1 \otimes \ldots \otimes h_n)\tau_n\big)_{k,\ell} = 0$ for $k < \ell$.

(2) Note that for $k \ge 1$ in the formula for $\big((h_1 \otimes \ldots h_n)\tau_n\big)_{k,\ell}$ in Definition 43 a summand is non-zero only if $n \le r'_0 + \big(\sum_{\beta=1}^{n} 1 + r'_\beta\big) = \ell$. Thus $\ell < n$ implies that $\big((h_1 \otimes \ldots \otimes h_n)\tau_n\big)_{k,\ell} = 0$ for $k \ge 1$, hence $\big((h_1 \otimes \ldots \otimes h_n)\tau_n\big)\pi_\ell = 0$. □

Remark 45 Suppose given morphisms of graded coalgebras $f, g \colon TA \to TB$ and $p \in \mathbf{Z}$.

(1) For an (f,g)-coderivation $h \colon TA \to TB$ of degree p we have $h\tau_1 = h$.

(2) The morphism $\tau_1 \colon \mathrm{Coder}(TA,TB)^{p,(f,g)} \to \mathsf{grHom}(TA,TB)^p$ is a split monomorphism.

Proof. (1) This follows from Lemma 37.

(2) By (1), $\tau_1 \colon \mathrm{Coder}(TA,TB)^{p,(f,g)} \to \mathsf{grHom}(TA,TB)^p$ is the inclusion map and hence split monic by Corollary 39. □

Lemma 46 *Let $n \ge 1$. Suppose given graded coalgebra morphisms $f_i \colon TA \to TB$ for $0 \le i \le n$ and (f_{i-1}, f_i)-coderivations $h_i \colon TA \to TB$ of degree p_i for $1 \le i \le n$. Then the* *following equation of graded linear maps from TA to $TB \otimes TB$ of degree $\sum_{i=1}^{n} p_i$ holds.*

$$
\begin{aligned}
((h_1 \otimes \ldots \otimes h_n)\tau_n)\Delta = \Delta\big(& f_0 \otimes (h_1 \otimes \ldots \otimes h_n)\tau_n \\
&+ \sum_{a=1}^{n-1} (h_1 \otimes \ldots \otimes h_a)\tau_a \otimes (h_{a+1} \otimes \ldots \otimes h_n)\tau_{n-a} \\
&+ (h_1 \otimes \ldots \otimes h_n)\tau_n \otimes f_n\big)
\end{aligned}
$$

Proof. It suffices to show that for $k, \ell_1, \ell_2 \geq 1$ we have

$$
\begin{aligned}
\iota_k\big((h_1 \otimes \ldots \otimes h_n)\tau_n\big)\Delta(\pi_{\ell_1} \otimes \pi_{\ell_2}) = \iota_k\Delta\big(& f_0 \otimes (h_1 \otimes \ldots \otimes h_n)\tau_n \\
&+ \sum_{a=1}^{n-1} (h_1 \otimes \ldots \otimes h_a)\tau_a \otimes (h_{a+1} \otimes \ldots \otimes h_n)\tau_{n-a} \\
&+ (h_1 \otimes \ldots \otimes h_n)\tau_n \otimes f_n\big)(\pi_{\ell_1} \otimes \pi_{\ell_2}). \qquad (*)
\end{aligned}
$$

Using Remark 20 the right-hand side equals the following.

$$
\begin{aligned}
&\iota_k\Delta\big(f_0 \otimes (h_1 \otimes \ldots \otimes h_n)\tau_n \\
&\qquad + \sum_{a=1}^{n-1} (h_1 \otimes \ldots \otimes h_a)\tau_a \otimes (h_{a+1} \otimes \ldots \otimes h_n)\tau_{n-a} \\
&\qquad + (h_1 \otimes \ldots \otimes h_n)\tau_n \otimes f_n\big)(\pi_{\ell_1} \otimes \pi_{\ell_2}) \\
&= \sum_{\substack{i+j=k \\ i,j\geq 1}} (\iota_i \otimes \iota_j)\big(f_0 \otimes (h_1 \otimes \ldots \otimes h_n)\tau_n\big)(\pi_{\ell_1} \otimes \pi_{\ell_2}) \\
&\qquad + \sum_{\substack{i+j=k \\ i,j\geq 1}} \sum_{a=1}^{n-1} (\iota_i \otimes \iota_j)\big((h_1 \otimes \ldots \otimes h_a)\tau_a \otimes (h_{a+1} \otimes \ldots \otimes h_n)\tau_{n-a}\big)(\pi_{\ell_1} \otimes \pi_{\ell_2}) \\
&\qquad + \sum_{\substack{i+j=k \\ i,j\geq 1}} (\iota_i \otimes \iota_j)\big((h_1 \otimes \ldots \otimes h_n)\tau_n \otimes f_n\big)(\pi_{\ell_1} \otimes \pi_{\ell_2}) \\
&= \sum_{\substack{i+j=k \\ i,j\geq 1}} (f_0)_{i,\ell_1} \otimes \big((h_1 \otimes \ldots \otimes h_n)\tau_n\big)_{j,\ell_2} \\
&\qquad + \sum_{a=1}^{n-1} \sum_{\substack{i+j=k \\ i,j\geq 1}} \big((h_1 \otimes \ldots \otimes h_a)\tau_a\big)_{i,\ell_1} \otimes \big((h_{a+1} \otimes \ldots \otimes h_n)\tau_{n-a}\big)_{j,\ell_2} \\
&\qquad + \sum_{\substack{i+j=k \\ i,j\geq 1}} \big((h_1 \otimes \ldots \otimes h_n)\tau_n\big)_{i,\ell_1} \otimes (f_n)_{j,\ell_2} \qquad (**)
\end{aligned}
$$

We proceed with the left-hand side of $(*)$, again using Remark 20 and Definition 43.

$$
\begin{aligned}
&\iota_k\big((h_1 \otimes \ldots \otimes h_n)\tau_n\big)\Delta(\pi_{\ell_1} \otimes \pi_{\ell_2}) \\
&\overset{\text{R 20}}{=} \iota_k\big((h_1 \otimes \ldots \otimes h_n)\tau_n\big)\pi_{\ell_1+\ell_2} \\
&= \big((h_1 \otimes \ldots \otimes h_n)\tau_n\big)_{k,\ell_1+\ell_2} \\
&\overset{\text{D 43}}{=} \sum_{\substack{r_0+\left(\sum_{\beta=1}^n s_\beta+r_\beta\right)=k \\ r'_0+\left(\sum_{\beta=1}^n 1+r'_\beta\right)=\ell_1+\ell_2 \\ r_0,\ldots,r_n,r'_0,\ldots,r'_n\geq 0,\, s_1,\ldots,s_n\geq 1}} (\hat{f}_0)_{r_0,r'_0} \otimes \bigotimes_{\beta=1}^n \left((h_\beta)_{s_\beta,1} \otimes (\hat{f}_\beta)_{r_\beta,r'_\beta}\right) \\
&= \sum_{\substack{r_0+\left(\sum_{\beta=1}^n s_\beta+r_\beta\right)=k \\ r'_0+\left(\sum_{\beta=1}^n 1+r'_\beta\right)=\ell_1+\ell_2 \\ \ell_1\leq r'_0 \\ r_0,\ldots,r_n,r'_0,\ldots,r'_n\geq 0,\, s_1,\ldots,s_n\geq 1}} (\hat{f}_0)_{r_0,r'_0} \otimes \bigotimes_{\beta=1}^n \left((h_\beta)_{s_\beta,1} \otimes (\hat{f}_\beta)_{r_\beta,r'_\beta}\right) \\
&\quad + \sum_{a=1}^{n-1} \sum_{\substack{r_0+\left(\sum_{\beta=1}^n s_\beta+r_\beta\right)=k \\ r'_0+\left(\sum_{\beta=1}^n 1+r'_\beta\right)=\ell_1+\ell_2 \\ r'_0+\left(\sum_{\beta=1}^{a-1} 1+r'_\beta\right)+1\leq\ell_1\leq r'_0+\left(\sum_{\beta=1}^{a} 1+r'_\beta\right) \\ r_0,\ldots,r_n,r'_0,\ldots,r'_n\geq 0,\, s_1,\ldots,s_n\geq 1}} (\hat{f}_0)_{r_0,r'_0} \otimes \bigotimes_{\beta=1}^n \left((h_\beta)_{s_\beta,1} \otimes (\hat{f}_\beta)_{r_\beta,r'_\beta}\right) \\
&\quad + \sum_{\substack{r_0+\left(\sum_{\beta=1}^n s_\beta+r_\beta\right)=k \\ r'_0+\left(\sum_{\beta=1}^n 1+r'_\beta\right)=\ell_1+\ell_2 \\ r'_0+\left(\sum_{\beta=1}^{n-1} 1+r'_\beta\right)+1\leq\ell_1 \\ r_0,\ldots,r_n,r'_0,\ldots,r'_n\geq 0,\, s_1,\ldots,s_n\geq 1}} (\hat{f}_0)_{r_0,r'_0} \otimes \bigotimes_{\beta=1}^n \left((h_\beta)_{s_\beta,1} \otimes (\hat{f}_\beta)_{r_\beta,r'_\beta}\right).
\end{aligned}
$$

We continue by considering the preceeding three summands separately. We make use of Remark 32.(4), Lemma 33 and Definition 43. We start with the first summand.

$$
\begin{aligned}
&\sum_{\substack{r_0+\left(\sum_{\beta=1}^n s_\beta+r_\beta\right)=k \\ r'_0+\left(\sum_{\beta=1}^n 1+r'_\beta\right)=\ell_1+\ell_2 \\ \ell_1\leq r'_0 \\ r_0,\ldots,r_n,r'_0,\ldots,r'_n\geq 0,\, s_1,\ldots,s_n\geq 1}} (\hat{f}_0)_{r_0,r'_0} \otimes \bigotimes_{\beta=1}^n \left((h_\beta)_{s_\beta,1} \otimes (\hat{f}_\beta)_{r_\beta,r'_\beta}\right) \\
&= \sum_{\substack{r_0+\left(\sum_{\beta=1}^n s_\beta+r_\beta\right)=k \\ u'_0+\left(\sum_{\beta=1}^n 1+r'_\beta\right)=\ell_2 \\ r_0,\ldots,r_n,u'_0,r'_1,\ldots,r'_n\geq 0,\, s_1,\ldots,s_n\geq 1}} (\hat{f}_0)_{r_0,\ell_1+u'_0} \otimes \bigotimes_{\beta=1}^n \left((h_\beta)_{s_\beta,1} \otimes (\hat{f}_\beta)_{r_\beta,r'_\beta}\right)
\end{aligned}
$$

$$\overset{\text{L 33}}{=} \sum_{\substack{r_0+\left(\sum_{\beta=1}^n s_\beta+r_\beta\right)=k\\ u_0'+\left(\sum_{\beta=1}^n 1+r_\beta'\right)=\ell_2\\ r_0,\dots,r_n,u_0',r_1',\dots,r_n'\geq 0,\, s_1,\dots,s_n\geq 1}} \sum_{\substack{i+u_0=r_0\\ i,u_0\geq 0}} (\hat{f}_0)_{i,\ell_1}\otimes(\hat{f}_0)_{u_0,u_0'}\otimes\bigotimes_{\beta=1}^n\Big((h_\beta)_{s_\beta,1}\otimes(\hat{f}_\beta)_{r_\beta,r_\beta'}\Big)$$

$$= \sum_{\substack{i+u_0+\left(\sum_{\beta=1}^n s_\beta+r_\beta\right)=k\\ u_0'+\left(\sum_{\beta=1}^n 1+r_\beta'\right)=\ell_2\\ i,u_0,r_1,\dots,r_n,u_0',r_1',\dots,r_n'\geq 0,\, s_1,\dots,s_n\geq 1}} (\hat{f}_0)_{i,\ell_1}\otimes(\hat{f}_0)_{u_0,u_0'}\otimes\bigotimes_{\beta=1}^n\Big((h_\beta)_{s_\beta,1}\otimes(\hat{f}_\beta)_{r_\beta,r_\beta'}\Big)$$

$$\overset{\text{R 32.(4)}}{=} \sum_{\substack{i+u_0+\left(\sum_{\beta=1}^n s_\beta+r_\beta\right)=k\\ u_0'+\left(\sum_{\beta=1}^n 1+r_\beta'\right)=\ell_2\\ u_0,r_1,\dots,r_n,u_0',r_1',\dots,r_n'\geq 0,\, i,s_1,\dots,s_n\geq 1}} (f_0)_{i,\ell_1}\otimes(\hat{f}_0)_{u_0,u_0'}\otimes\bigotimes_{\beta=1}^n\Big((h_\beta)_{s_\beta,1}\otimes(\hat{f}_\beta)_{r_\beta,r_\beta'}\Big)$$

$$= \sum_{\substack{i+j=k\\ i,j\geq 1}} \sum_{\substack{u_0+\left(\sum_{\beta=1}^n s_\beta+r_\beta\right)=j\\ u_0'+\left(\sum_{\beta=1}^n 1+r_\beta'\right)=\ell_2\\ u_0,r_1,\dots,r_n,u_0',r_1',\dots,r_n'\geq 0,\, s_1,\dots,s_n\geq 1}} (f_0)_{i,\ell_1}\otimes(\hat{f}_0)_{u_0,u_0'}\otimes\bigotimes_{\beta=1}^n\Big((h_\beta)_{s_\beta,1}\otimes(\hat{f}_\beta)_{r_\beta,r_\beta'}\Big)$$

$$\overset{\text{D 43}}{=} \sum_{\substack{i+j=k\\ i,j\geq 1}} (f_0)_{i,\ell_1}\otimes\big((h_1\otimes\ldots\otimes h_n)\tau_n\big)_{j,\ell_2}$$

We proceed with the second summand, for $1\leq a\leq n-1$.

$$\sum_{\substack{r_0+\left(\sum_{\beta=1}^n s_\beta+r_\beta\right)=k\\ r_0'+\left(\sum_{\beta=1}^n 1+r_\beta'\right)=\ell_1+\ell_2\\ r_0'+\left(\sum_{\beta=1}^{a-1} 1+r_\beta'\right)+1\leq\ell_1\leq r_0'+\left(\sum_{\beta=1}^{a} 1+r_\beta'\right)\\ r_0,\dots,r_n,r_0',\dots,r_n'\geq 0,\, s_1,\dots,s_n\geq 1}} (\hat{f}_0)_{r_0,r_0'}\otimes\bigotimes_{\beta=1}^n\Big((h_\beta)_{s_\beta,1}\otimes(\hat{f}_\beta)_{r_\beta,r_\beta'}\Big)$$

$$= \sum_{\substack{\left(\sum_{\beta=1}^a r_{\beta-1}+s_\beta\right)+r_a+\left(\sum_{\beta=a+1}^n s_\beta+r_\beta\right)=k\\ \left(\sum_{\beta=1}^a r_{\beta-1}'+1\right)+r_a'+\left(\sum_{\beta=a+1}^n 1+r_\beta'\right)=\ell_1+\ell_2\\ \left(\sum_{\beta=1}^a r_{\beta-1}'+1\right)\leq\ell_1\leq\left(\sum_{\beta=1}^a r_{\beta-1}'+1\right)+r_a'\\ r_0,\dots,r_n,r_0',\dots,r_n'\geq 0,\, s_1,\dots,s_n\geq 1}}$$

$$\bigotimes_{\beta=1}^a\Big((\hat{f}_{\beta-1})_{r_{\beta-1},r_{\beta-1}'}\otimes(h_\beta)_{s_\beta,1}\Big)\otimes(\hat{f}_a)_{r_a,r_a'}\otimes\bigotimes_{\beta=a+1}^n\Big((h_\beta)_{s_\beta,1}\otimes(\hat{f}_\beta)_{r_\beta,r_\beta'}\Big)$$

$$= \sum_{\substack{\left(\sum_{\beta=1}^a r_{\beta-1}+s_\beta\right)+r_a+\left(\sum_{\beta=a+1}^n s_\beta+r_\beta\right)=k\\ \left(\sum_{\beta=1}^a r_{\beta-1}'+1\right)+r_a'+\left(\sum_{\beta=a+1}^n 1+r_\beta'\right)=\ell_1+\ell_2\\ \left(\sum_{\beta=1}^a r_{\beta-1}'+1\right)\leq\ell_1,\ \left(\sum_{\beta=a+1}^n 1+r_\beta'\right)\leq\ell_2\\ r_0,\dots,r_n,r_0',\dots,r_n'\geq 0,\, s_1,\dots,s_n\geq 1}}$$

$$\bigotimes_{\beta=1}^a\Big((\hat{f}_{\beta-1})_{r_{\beta-1},r_{\beta-1}'}\otimes(h_\beta)_{s_\beta,1}\Big)\otimes(\hat{f}_a)_{r_a,r_a'}\otimes\bigotimes_{\beta=a+1}^n\Big((h_\beta)_{s_\beta,1}\otimes(\hat{f}_\beta)_{r_\beta,r_\beta'}\Big)$$

$$
\begin{aligned}
&= \sum_{\substack{\left(\sum_{\beta=1}^{a} r_{\beta-1}+s_\beta\right)+r_a+\left(\sum_{\beta=a+1}^{n} s_\beta+r_\beta\right)=k \\ \left(\sum_{\beta=1}^{a} r'_{\beta-1}+1\right)+u'_a=\ell_1,\ v'_a+\left(\sum_{\beta=a+1}^{n} 1+r'_\beta\right)=\ell_2 \\ r_0,\dots,r_n,r'_0,\dots,r'_{a-1},u'_a,v'_a,r'_{a+1},\dots,r'_n\geq 0,\ s_1,\dots,s_n\geq 1}} \\
&\quad \bigotimes_{\beta=1}^{a}\left((\hat{f}_{\beta-1})_{r_{\beta-1},r'_{\beta-1}}\otimes (h_\beta)_{s_\beta,1}\right)\otimes (\hat{f}_a)_{r_a,u'_a+v'_a}\otimes \bigotimes_{\beta=a+1}^{n}\left((h_\beta)_{s_\beta,1}\otimes (\hat{f}_\beta)_{r_\beta,r'_\beta}\right) \\
&\overset{\text{L 33}}{=} \sum_{\substack{\left(\sum_{\beta=1}^{a} r_{\beta-1}+s_\beta\right)+r_a+\left(\sum_{\beta=a+1}^{n} s_\beta+r_\beta\right)=k \\ \left(\sum_{\beta=1}^{a} r'_{\beta-1}+1\right)+u'_a=\ell_1,\ v'_a+\left(\sum_{\beta=a+1}^{n} 1+r'_\beta\right)=\ell_2 \\ r_0,\dots,r_n,r'_0,\dots,r'_{a-1},u'_a,v'_a,r'_{a+1},\dots,r'_n\geq 0,\ s_1,\dots,s_n\geq 1}} \sum_{\substack{u_a+v_a=r_a \\ u_a,v_a\geq 0}} \\
&\quad \bigotimes_{\beta=1}^{a}\left((\hat{f}_{\beta-1})_{r_{\beta-1},r'_{\beta-1}}\otimes (h_\beta)_{s_\beta,1}\right)\otimes (\hat{f}_a)_{u_a,u'_a}\otimes (\hat{f}_a)_{v_a,v'_a}\otimes \bigotimes_{\beta=a+1}^{n}\left((h_\beta)_{s_\beta,1}\otimes (\hat{f}_\beta)_{r_\beta,r'_\beta}\right) \\
&= \sum_{\substack{\left(\sum_{\beta=1}^{a} r_{\beta-1}+s_\beta\right)+u_a+v_a+\left(\sum_{\beta=a+1}^{n} s_\beta+r_\beta\right)=k \\ \left(\sum_{\beta=1}^{a} r'_{\beta-1}+1\right)+u'_a=\ell_1,\ v'_a+\left(\sum_{\beta=a+1}^{n} 1+r'_\beta\right)=\ell_2 \\ r_0,\dots,r_{a-1},u_a,v_a,r_{a+1},\dots,r_n,r'_0,\dots,r'_{a-1},u'_a,v'_a,r'_{a+1},\dots,r'_n\geq 0,\ s_1,\dots,s_n\geq 1}} \\
&\quad \bigotimes_{\beta=1}^{a}\left((\hat{f}_{\beta-1})_{r_{\beta-1},r'_{\beta-1}}\otimes (h_\beta)_{s_\beta,1}\right)\otimes (\hat{f}_a)_{u_a,u'_a}\otimes (\hat{f}_a)_{v_a,v'_a}\otimes \bigotimes_{\beta=a+1}^{n}\left((h_\beta)_{s_\beta,1}\otimes (\hat{f}_\beta)_{r_\beta,r'_\beta}\right) \\
&= \sum_{\substack{i+j=k \\ i,j\geq 1}} \sum_{\substack{\left(\sum_{\beta=1}^{a} r_{\beta-1}+s_\beta\right)+u_a=i,\ v_a+\left(\sum_{\beta=a+1}^{n} s_\beta+r_\beta\right)=j \\ \left(\sum_{\beta=1}^{a} r'_{\beta-1}+1\right)+u'_a=\ell_1,\ v'_a+\left(\sum_{\beta=a+1}^{n} 1+r'_\beta\right)=\ell_2 \\ r_0,\dots,r_{a-1},u_a,v_a,r_{a+1},\dots,r_n,r'_0,\dots,r'_{a-1},u'_a,v'_a,r'_{a+1},\dots,r'_n\geq 0,\ s_1,\dots,s_n\geq 1}} \\
&\quad \bigotimes_{\beta=1}^{a}\left((\hat{f}_{\beta-1})_{r_{\beta-1},r'_{\beta-1}}\otimes (h_\beta)_{s_\beta,1}\right)\otimes (\hat{f}_a)_{u_a,u'_a}\otimes (\hat{f}_a)_{v_a,v'_a}\otimes \bigotimes_{\beta=a+1}^{n}\left((h_\beta)_{s_\beta,1}\otimes (\hat{f}_\beta)_{r_\beta,r'_\beta}\right) \\
&= \sum_{\substack{i+j=k \\ i,j\geq 1}} \sum_{\substack{\left(\sum_{\beta=1}^{a} r_{\beta-1}+s_\beta\right)+u_a=i \\ \left(\sum_{\beta=1}^{a} r'_{\beta-1}+1\right)+u'_a=\ell_1 \\ r_0,\dots,r_{a-1},u_a,r'_0,\dots,r'_{a-1},u'_a\geq 0,\ s_1,\dots,s_a\geq 1}} \sum_{\substack{v_a+\left(\sum_{\beta=a+1}^{n} s_\beta+r_\beta\right)=j \\ v'_a+\left(\sum_{\beta=a+1}^{n} 1+r'_\beta\right)=\ell_2 \\ v_a,r_{a+1},\dots,r_n,v'_a,r'_{a+1},\dots,r'_n\geq 0,\ s_{a+1},\dots,s_n\geq 1}} \\
&\quad \bigotimes_{\beta=1}^{a}\left((\hat{f}_{\beta-1})_{r_{\beta-1},r'_{\beta-1}}\otimes (h_\beta)_{s_\beta,1}\right)\otimes (\hat{f}_a)_{u_a,u'_a}\otimes (\hat{f}_a)_{v_a,v'_a}\otimes \bigotimes_{\beta=a+1}^{n}\left((h_\beta)_{s_\beta,1}\otimes (\hat{f}_\beta)_{r_\beta,r'_\beta}\right) \\
&\overset{\text{D 43}}{=} \sum_{\substack{i+j=k \\ i,j\geq 1}} \left((h_1\otimes\ldots\otimes h_a)\tau_a\right)_{i,\ell_1}\otimes\left((h_{a+1}\otimes\ldots\otimes h_n)\tau_{n-a}\right)_{j,\ell_2}
\end{aligned}
$$

We still have to consider the last summand.

$$
\sum_{\substack{r_0+\left(\sum_{\beta=1}^{n} s_\beta+r_\beta\right)=k \\ r'_0+\left(\sum_{\beta=1}^{n} 1+r'_\beta\right)=\ell_1+\ell_2 \\ r'_0+\left(\sum_{\beta=1}^{n-1} 1+r'_\beta\right)+1\leq \ell_1 \\ r_0,\dots,r_n,r'_0,\dots,r'_n\geq 0,\ s_1,\dots,s_n\geq 1}} (\hat{f}_0)_{r_0,r'_0}\otimes \bigotimes_{\beta=1}^{n}\left((h_\beta)_{s_\beta,1}\otimes (\hat{f}_\beta)_{r_\beta,r'_\beta}\right)
$$

$$= \sum_{\substack{\left(\sum_{\beta=1}^{n} r_{\beta-1}+s_\beta\right)+r_n=k \\ \left(\sum_{\beta=1}^{n} r'_{\beta-1}+1\right)+r'_n=\ell_1+\ell_2 \\ \ell_2\leq r'_n \\ r_0,\dots,r_n,r'_0,\dots,r'_n\geq 0,\, s_1,\dots,s_n\geq 1}} \bigotimes_{\beta=1}^{n} \Big((\hat{f}_{\beta-1})_{r_{\beta-1},r'_{\beta-1}} \otimes (h_\beta)_{s_\beta,1} \Big) \otimes (\hat{f}_n)_{r_n,r'_n}$$

$$= \sum_{\substack{\left(\sum_{\beta=1}^{n} r_{\beta-1}+s_\beta\right)+r_n=k \\ \left(\sum_{\beta=1}^{n} r'_{\beta-1}+1\right)+u'_n=\ell_1 \\ r_0,\dots,r_n,r'_0,\dots,r'_{n-1},u'_n\geq 0,\, s_1,\dots,s_n\geq 1}} \bigotimes_{\beta=1}^{n} \Big((\hat{f}_{\beta-1})_{r_{\beta-1},r'_{\beta-1}} \otimes (h_\beta)_{s_\beta,1} \Big) \otimes (\hat{f}_n)_{r_n,u'_n+\ell_2}$$

$$\overset{\text{L 33}}{=} \sum_{\substack{\left(\sum_{\beta=1}^{n} r_{\beta-1}+s_\beta\right)+r_n=k \\ \left(\sum_{\beta=1}^{n} r'_{\beta-1}+1\right)+u'_n=\ell_1 \\ r_0,\dots,r_n,r'_0,\dots,r'_{n-1},u'_n\geq 0,\, s_1,\dots,s_n\geq 1}} \sum_{\substack{u_n+j=r_n \\ u_n,\, j\geq 0}} \bigotimes_{\beta=1}^{n} \Big((\hat{f}_{\beta-1})_{r_{\beta-1},r'_{\beta-1}} \otimes (h_\beta)_{s_\beta,1} \Big) \otimes (\hat{f}_n)_{u_n,u'_n} \otimes (\hat{f}_n)_{j,\ell_2}$$

$$= \sum_{\substack{\left(\sum_{\beta=1}^{n} r_{\beta-1}+s_\beta\right)+u_n+j=k \\ \left(\sum_{\beta=1}^{n} r'_{\beta-1}+1\right)+u'_n=\ell_1 \\ j,r_0,\dots,r_{n-1},u_n,r'_0,\dots,r'_{n-1},u'_n\geq 0,\, s_1,\dots,s_n\geq 1}} \bigotimes_{\beta=1}^{n} \Big((\hat{f}_{\beta-1})_{r_{\beta-1},r'_{\beta-1}} \otimes (h_\beta)_{s_\beta,1} \Big) \otimes (\hat{f}_n)_{u_n,u'_n} \otimes (\hat{f}_n)_{j,\ell_2}$$

$$\overset{\text{R 32.(4)}}{=} \sum_{\substack{\left(\sum_{\beta=1}^{n} r_{\beta-1}+s_\beta\right)+u_n+j=k \\ \left(\sum_{\beta=1}^{n} r'_{\beta-1}+1\right)+u'_n=\ell_1 \\ r_0,\dots,r_{n-1},u_n,r'_0,\dots,r'_{n-1},u'_n\geq 0,\, j,s_1,\dots,s_n\geq 1}} \bigotimes_{\beta=1}^{n} \Big((\hat{f}_{\beta-1})_{r_{\beta-1},r'_{\beta-1}} \otimes (h_\beta)_{s_\beta,1} \Big) \otimes (\hat{f}_n)_{u_n,u'_n} \otimes (f_n)_{j,\ell_2}$$

$$= \sum_{\substack{i+j=k \\ i,j\geq 1}} \sum_{\substack{\left(\sum_{\beta=1}^{n} r_{\beta-1}+s_\beta\right)+u_n=i \\ \left(\sum_{\beta=1}^{n} r'_{\beta-1}+1\right)+u'_n=\ell_1 \\ r_0,\dots,r_{n-1},u_n,r'_0,\dots,r'_{n-1},u'_n\geq 0,\, s_1,\dots,s_n\geq 1}} \bigotimes_{\beta=1}^{n} \Big((\hat{f}_{\beta-1})_{r_{\beta-1},r'_{\beta-1}} \otimes (h_\beta)_{s_\beta,1} \Big) \otimes (\hat{f}_n)_{u_n,u'_n} \otimes (f_n)_{j,\ell_2}$$

$$\overset{\text{D 43}}{=} \sum_{\substack{i+j=k \\ i,j\geq 1}} \big((h_1 \otimes \ldots \otimes h_n)\tau_n\big)_{i,\ell_1} \otimes (f_n)_{j,\ell_2}$$

Comparing the results of these three calculations with $(**)$ shows that $(*)$ holds true. $\square$

Definition 47 Let A and B be graded modules. Suppose given differential graded tensor coalgebras (TA, Δ, m) and (TB, Δ, m), cf. Definition 29.

Given $k \geq 1$, the graded linear map τ_k from Definition 43 defines a $\mathcal{Z}_{TA,TB}$-graded linear map

$$t_k\colon \quad \mathrm{Coder}(TA, TB)^{\otimes k} \quad\longrightarrow\quad \mathrm{PreCoder}(TA, TB)$$

with

$$(h_1 \otimes \ldots \otimes h_k)t_k^{p,(f_0,f_k)} := (h_1 \otimes \ldots \otimes h_k)\tau_k$$

for $f_0, \dots, f_k \in \mathsf{dgCoalg}(TA, TB)$, $p_0, \dots, p_k \in \mathbf{Z}$ and (f_{i-1}, f_i)-coderivations $h_i\colon TA \to TB$ of degree p_i for $1 \leq i \leq k$ and $p := \sum_{i=1}^{k} p_i$.

By Lemma 22 the tuple $(t_k)_{k\geq 1}$ defines a morphism of $\mathcal{Z}_{TA,TB}$-graded coalgebras

$$\mathfrak{t}\colon\quad T\,\mathrm{Coder}(TA,TB) \;\longrightarrow\; T\,\mathrm{PreCoder}(TA,TB)$$

with $\mathfrak{t}_{k,1} := t_k$.

In Theorem 49 we will construct a differential on $T\,\mathrm{Coder}(TA,TB)$ such that $\mathfrak{t}$ becomes a morphism of differential $\mathcal{Z}_{TA,TB}$-graded coalgebras, where $T\,\mathrm{PreCoder}(TA,TB)$ is endowed with the differential $\mathfrak{m}$ from Lemma 42.

Lemma 48 *The morphism of $\mathcal{Z}_{TA,TB}$-graded coalgebras*

$$\mathfrak{t}\colon\quad T\,\mathrm{Coder}(TA,TB) \;\longrightarrow\; T\,\mathrm{PreCoder}(TA,TB)$$

from Definition 47 is injective.

Proof. Given $p \in \mathbf{Z}$ and $f, g \in \mathsf{dgCoalg}(TA,TB)$, we have $\mathfrak{t}_{1,1}^{p,(f,g)} = \tau_1$. By Remark 45.(2) the graded linear map $\tau_1\colon\ \mathrm{Coder}(TA,TB)^{p,(f,g)} \to \mathrm{PreCoder}(TA,TB)^{p,(f,g)}$ is a split monomorphism, hence $\mathfrak{t}_{1,1}$ is a split monomorphism. Therefore $\mathfrak{t}$ is injective by Lemma 25. □

2.1.4 The A_∞-category of coderivations

Let A and B be graded modules.

Suppose we are given differential graded tensor coalgebras (TA, Δ, m) and (TB, Δ, m), cf. Definition 18.

Recall the $\mathcal{Z}_{TA,TB}$-graded module of precoderivations $\mathrm{PreCoder}(TA,TB)$ and the $\mathcal{Z}_{TA,TB}$-graded module of coderivations $\mathrm{Coder}(TA,TB)$, cf. Definition 41.

Recall the differential $\mathfrak{m}$ on the tensor coalgebra $(T\,\mathrm{PreCoder}(TA,TB), \Delta)$ that makes $(T\,\mathrm{PreCoder}(TA,TB), \Delta, \mathfrak{m})$ into a differential $\mathcal{Z}_{TA,TB}$-graded coalgebra, cf. Lemma 42.

Recall the morphism of $\mathcal{Z}_{TA,TB}$-graded coalgebras $\mathfrak{t}\colon T\,\mathrm{Coder}(TA,TB) \to T\,\mathrm{PreCoder}(TA,TB)$ between the tensor coalgebras over $\mathrm{Coder}(TA,TB)$ and $T\,\mathrm{PreCoder}(TA,TB)$, cf. Definition 47.

Theorem 49 *There is a uniquely determined coderivation*

$$M\colon\quad T\,\mathrm{Coder}(TA,TB) \;\longrightarrow\; T\,\mathrm{Coder}(TA,TB)$$

such that $M\mathfrak{t} = \mathfrak{t}\mathfrak{m}$ and such that $(T\,\mathrm{Coder}(TA,TB), \Delta, M)$ is a differential $\mathcal{Z}_{TA,TB}$-graded coalgebra.

$$\begin{array}{ccc} T\,\mathrm{Coder}(TA,TB) & \xrightarrow{M} & T\,\mathrm{Coder}(TA,TB) \\ \downarrow{\scriptstyle\mathfrak{t}} & & \downarrow{\scriptstyle\mathfrak{t}} \\ T\,\mathrm{PreCoder}(TA,TB) & \xrightarrow{\mathfrak{m}} & T\,\mathrm{PreCoder}(TA,TB) \end{array}$$

I.e. $\mathfrak{t}$ is a morphism of differential $\mathcal{Z}_{TA,TB}$-graded coalgebras between $(T\,\mathrm{Coder}(TA,TB), \Delta, M)$ and $(T\,\mathrm{PreCoder}(TA,TB), \Delta, \mathfrak{m})$.

In particular, the following formulas hold.

$$M_{1,1}\mathfrak{t}_{1,1} = \mathfrak{t}_{1,1}\mathfrak{m}_{1,1} \quad \textit{and} \quad M_{2,1}\mathfrak{t}_{1,1} = \mathfrak{t}_{2,1}\mathfrak{m}_{1,1} - (\mathrm{id}\otimes M_{1,1} + M_{1,1}\otimes\mathrm{id})\mathfrak{t}_{2,1}$$

Proof. Uniqueness. Suppose also $\tilde{M}\colon T\,\mathrm{Coder}(TA,TB) \to T\,\mathrm{Coder}(TA,TB)$ is a $\mathcal{Z}_{TA,TB}$-coderivation with $\tilde{M}\mathfrak{t} = \mathfrak{t}\mathfrak{m}$. Then $\tilde{M}\mathfrak{t} = M\mathfrak{t}$. Since $\mathfrak{t}$ is injective by Lemma 48, this implies $\tilde{M} = M$.

Existence. We *claim* that for $k \geq 1$ there exist $\mathcal{Z}_{TA,TB}$-graded linear maps

$$\mathfrak{M}_k\colon \quad \mathrm{Coder}(TA,TB)^{\otimes k} \longrightarrow \mathrm{Coder}(TA,TB)$$

of degree 1 such that

$$\begin{aligned} 0 \stackrel{!}{=} \mathfrak{t}_{k,1}\mathfrak{m}_{1,1} - \sum_{i=1}^{k} \sum_{\substack{r+s+t=k \\ r+1+t=i \\ r,t\geq 0,\, s\geq 1}} (\mathrm{id}^{\otimes r} \otimes \mathfrak{M}_s \otimes \mathrm{id}^{\otimes t})\mathfrak{t}_{i,1} \qquad (*_k) \\ = \mathfrak{t}_{k,1}\mathfrak{m}_{1,1} - \sum_{i=1}^{k} \sum_{\substack{r+t=i-1 \\ r,t\geq 0}} (\mathrm{id}^{\otimes r} \otimes \mathfrak{M}_{k-i+1} \otimes \mathrm{id}^{\otimes t})\mathfrak{t}_{i,1} \end{aligned}$$

holds. Note that only $\mathfrak{M}_s$ with $s \leq k$ appear in this equation.

We prove the claim by induction on k.

For $k = 1$, suppose given $p \in \mathbf{Z}$ and $f, g \in \mathsf{dgCoalg}(TA,TB)$ and an (f,g)-coderivation $h\colon TA \to TB$ of degree p. Recall that $\mathfrak{t}_{1,1}^{p,(f,g)} = \tau_1$ by Definition 47 and thus by Remark 45.(1) the morphism $\mathfrak{t}_{1,1}\colon \mathrm{Coder}(TA,TB) \to \mathrm{PreCoder}(TA,TB)$ is the degreewise inclusion. Recall from Lemma 42 that $\mathfrak{m}_{1,1}^{p,(f,g)} = \mu^p$ with the differential μ from Lemma 40. We have using Lemma 40.(2)

$$\begin{aligned} (h\mathfrak{t}_{1,1}^{p,(f,g)}\mathfrak{m}_{1,1}^{p,(f,g)})\Delta &= (h\mu^p)\Delta \\ &\stackrel{\mathrm{L}\,40.(2)}{=} \Delta(f \otimes h\mu^p + (-1)^p f\mu^0 \otimes h + h \otimes g\mu^0 + h\mu^p \otimes g) \\ &= \Delta(f \otimes h\mu^p + h\mu^p \otimes g) \\ &= \Delta(f \otimes (h\mathfrak{t}_{1,1}^{p,(f,g)}\mathfrak{m}_{1,1}^{p,(f,g)}) + (h\mathfrak{t}_{1,1}^{p,(f,g)}\mathfrak{m}_{1,1}^{p,(f,g)}) \otimes g) \end{aligned}$$

Here we used that $f\mu^0 = fm - mf = 0$ since f is a morphism of differential graded coalgebras. Similarly, we have $g\mu^0 = 0$. It follows that $h\mathfrak{t}_{1,1}^{p,(f,g)}\mathfrak{m}_{1,1}^{p,(f,g)}$ is again an (f,g)-coderivation. Thus there is a $\mathcal{Z}_{TA,TB}$-graded linear map $\mathfrak{M}_1\colon \mathrm{Coder}(TA,TB) \to \mathrm{PreCoder}(TA,TB)$ of degree 1 such that $\mathfrak{t}_{1,1}\mathfrak{m}_{1,1} - \mathfrak{M}_1\mathfrak{t}_{1,1} = 0$.

Now let $k > 1$ and suppose that the $\mathcal{Z}_{TA,TB}$-graded linear maps $\mathfrak{M}_\ell$ have already been constructed such that $(*_\ell)$ holds for $\ell < k$.

We have to show that there is a $\mathcal{Z}_{TA,TB}$-graded linear map

$$\mathfrak{M}_k\colon \mathrm{Coder}(TA,TB)^{\otimes k} \to \mathrm{Coder}(TA,TB)$$

of degree 1 such that $(*_k)$ holds. Consider

$$\tilde{\mathfrak{M}}_k := \mathfrak{t}_{k,1}\mathfrak{m}_{1,1} - \sum_{i=2}^{k} \sum_{\substack{r+t=i-1 \\ r,t\geq 0}} (\mathrm{id}^{\otimes r} \otimes \mathfrak{M}_{k-i+1} \otimes \mathrm{id}^{\otimes t})\mathfrak{t}_{i,1}.$$

Suppose given $p_1, \dots, p_k \in \mathbf{Z}$, $f_0, \dots, f_k \in \mathsf{dgCoalg}(TA, TB)$ and (f_{i-1}, f_i)-coderivations $h_i \colon TA \to TB$ of degree p_i for $1 \leq i \leq k$. Let $p := \sum_{i=1}^{k} p_i$.
We show that $(h_1 \otimes \ldots \otimes h_k)\tilde{\mathfrak{M}}_k^{p,(f_0,f_k)}$ is an (f_0, f_k)-coderivation of degree $p+1$.
Given $1 \leq i \leq j \leq k$, we write $h^{\otimes}_{[i,j]} := h_i \otimes h_{i+1} \otimes \ldots \otimes h_j$ and $h^{\otimes}_{[i+1,i]} := \mathrm{id}_{\dot{R}}$ for $0 \leq i \leq k-1$.
Recall that we sometimes omit the degrees on graded linear maps, e.g. we write $\mathfrak{M}_k := \mathfrak{M}_k^{p,(f_0,f_k)}$.
Consider

$$
\begin{aligned}
&((h_1 \otimes \ldots \otimes h_k)\tilde{\mathfrak{M}}_k)\Delta \\
&\quad = ((h_1 \otimes \ldots \otimes h_k)\mathfrak{t}_{k,1}\mathfrak{m}_{1,1})\Delta \\
&\quad\quad - \sum_{i=2}^{k} \sum_{\substack{r+t=i-1 \\ r,t \geq 0}} (-1)^{\sum_{\beta=k-t+1}^{k} p_\beta} \Big((h^{\otimes}_{[1,r]} \otimes h^{\otimes}_{[r+1,r+k-i+1]}\mathfrak{M}_{k-i+1} \otimes h^{\otimes}_{[k-t+1,k]})\mathfrak{t}_{i,1} \Big)\Delta. \qquad (**)
\end{aligned}
$$

We proceed with the first summand in $(**)$. Using Lemma 42 and Definition 47 we have

$$((h_1 \otimes \ldots \otimes h_k)\mathfrak{t}_{k,1}\mathfrak{m}_{1,1})\Delta = ((h_1 \otimes \ldots \otimes h_k)\tau_k\mu)\Delta.$$

By Lemma 46 we have

$$((h_1 \otimes \ldots \otimes h_k)\tau_k)\Delta = \Delta\Big(f_0 \otimes h^{\otimes}_{[1,k]}\tau_k + \Big(\sum_{\substack{a+b=k \\ a,b \geq 1}} h^{\otimes}_{[1,a]}\tau_a \otimes h^{\otimes}_{[k-b+1,k]}\tau_b \Big) + h^{\otimes}_{[1,k]}\tau_k \otimes f_k \Big).$$

Hence we can apply Lemma 40.(2) and obtain

$$
\begin{aligned}
&((h_1 \otimes \ldots \otimes h_k)\tau_k\mu)\Delta \\
&\quad = \Delta\Big(f_0 \otimes h^{\otimes}_{[1,k]}\tau_k\mu + (-1)^p f_0\mu \otimes h^{\otimes}_{[1,k]}\tau_k \\
&\quad\quad + \Big(\sum_{\substack{a+b=k \\ a,b \geq 1}} h^{\otimes}_{[1,a]}\tau_a \otimes h^{\otimes}_{[k-b+1,k]}\tau_b\mu \Big) + \Big(\sum_{\substack{a+b=k \\ a,b \geq 1}} (-1)^{\sum_{\beta=k-b+1}^{k} p_\beta} h^{\otimes}_{[1,a]}\tau_a\mu \otimes h^{\otimes}_{[k-b+1,k]}\tau_b \Big) \\
&\quad\quad + h^{\otimes}_{[1,k]}\tau_k \otimes f_k\mu + h^{\otimes}_{[1,k]}\tau_k\mu \otimes f_k \Big) \\
&\quad = \Delta\Big(f_0 \otimes h^{\otimes}_{[1,k]}\tau_k\mu + \Big(\sum_{\substack{a+b=k \\ a,b \geq 1}} h^{\otimes}_{[1,a]}\tau_a \otimes h^{\otimes}_{[k-b+1,k]}\tau_b\mu \Big) \\
&\quad\quad + \Big(\sum_{\substack{a+b=k \\ a,b \geq 1}} (-1)^{\sum_{\beta=k-b+1}^{k} p_\beta} h^{\otimes}_{[1,a]}\tau_a\mu \otimes h^{\otimes}_{[k-b+1,k]}\tau_b \Big) + h^{\otimes}_{[1,k]}\tau_k\mu \otimes f_k \Big) \\
&\quad = \Delta\Big(f_0 \otimes h^{\otimes}_{[1,k]}\mathfrak{t}_{k,1}\mathfrak{m}_{1,1} + \Big(\sum_{\substack{a+b=k \\ a,b \geq 1}} h^{\otimes}_{[1,a]}\mathfrak{t}_{a,1} \otimes h^{\otimes}_{[k-b+1,k]}\mathfrak{t}_{b,1}\mathfrak{m}_{1,1} \Big) \\
&\quad\quad + \Big(\sum_{\substack{a+b=k \\ a,b \geq 1}} (-1)^{\sum_{\beta=k-b+1}^{k} p_\beta} h^{\otimes}_{[1,a]}\mathfrak{t}_{a,1}\mathfrak{m}_{1,1} \otimes h^{\otimes}_{[k-b+1,k]}\mathfrak{t}_{b,1} \Big) + h^{\otimes}_{[1,k]}\mathfrak{t}_{k,1}\mathfrak{m}_{1,1} \otimes f_k \Big)
\end{aligned}
$$

$$(***)$$

Here we used Lemma 40.(1) to conclude that $f\mu = fm - mf = 0$ for morphisms of differential graded coalgebras $f \colon TA \to TB$. Moreover, in the last step we made use of Lemma 42 and Definition 47.

We continue with the second summand in $(**)$. Note that by the induction hypothesis $h^{\otimes}_{[r+1,r+k-i+1]}\mathfrak{M}_{k-i+1}$ is an $(f_r, f_{r+k-i+1})$-coderivation for $2 \leq i \leq k$. Hence we can apply Lemma 46 and obtain

$$\begin{aligned}
&\sum_{i=2}^{k} \sum_{\substack{r+t=i-1\\ r,t\geq 0}} (-1)^{\sum_{\beta=k-t+1}^{k} p_\beta} \Big((h^{\otimes}_{[1,r]} \otimes h^{\otimes}_{[r+1,r+k-i+1]}\mathfrak{M}_{k-i+1} \otimes h^{\otimes}_{[k-t+1,k]})\mathfrak{t}_{i,1}\Big)\Delta \\
&= \sum_{i=2}^{k} \sum_{\substack{r+t=i-1\\ r,t\geq 0}} (-1)^{\sum_{\beta=k-t+1}^{k} p_\beta} \Big((h^{\otimes}_{[1,r]} \otimes h^{\otimes}_{[r+1,r+k-i+1]}\mathfrak{M}_{k-i+1} \otimes h^{\otimes}_{[k-t+1,k]})\tau_i\Big)\Delta \\
&= \sum_{i=2}^{k} \sum_{\substack{r+t=i-1\\ r,t\geq 0}} (-1)^{\sum_{\beta=k-t+1}^{k} p_\beta} \Delta\Bigg(f_0 \otimes (h^{\otimes}_{[1,r]} \otimes h^{\otimes}_{[r+1,r+k-i+1]}\mathfrak{M}_{k-i+1} \otimes h^{\otimes}_{[k-t+1,k]})\tau_i \\
&\qquad + \Big(\sum_{a'=r+1}^{i} (h^{\otimes}_{[1,r]} \otimes h^{\otimes}_{[r+1,r+k-i+1]}\mathfrak{M}_{k-i+1} \otimes h^{\otimes}_{[k-t+1,k-i+a']})\tau_{a'} \otimes h^{\otimes}_{[k-i+a'+1,k]}\tau_{i-a'}\Big) \\
&\qquad + \Big(\sum_{a=1}^{r} h^{\otimes}_{[1,a]}\tau_a \otimes (h^{\otimes}_{[a+1,r]} \otimes h^{\otimes}_{[r+1,r+k-i+1]}\mathfrak{M}_{k-i+1} \otimes h^{\otimes}_{[k-t+1,k]})\tau_{i-a}\Big) \\
&\qquad + (h^{\otimes}_{[1,r]} \otimes h^{\otimes}_{[r+1,r+k-i+1]}\mathfrak{M}_{k-i+1} \otimes h^{\otimes}_{[k-t+1,k]})\tau_i \otimes f_k \Bigg) \\
&= \Delta\Bigg(f_0 \otimes \Big(\sum_{i=2}^{k} \sum_{\substack{r+t=i-1\\ r,t\geq 0}} (-1)^{\sum_{\beta=k-t+1}^{k} p_\beta} \Big((h^{\otimes}_{[1,r]} \otimes h^{\otimes}_{[r+1,r+k-i+1]}\mathfrak{M}_{k-i+1} \otimes h^{\otimes}_{[k-t+1,k]})\tau_i\Big) \\
&\quad + \sum_{i=2}^{k} \sum_{\substack{r+t=i-1\\ r,t\geq 0}} \sum_{\substack{a'+b=i\\ a'\geq r+1,\, b\geq 1}} \\
&\quad (-1)^{\sum_{\beta=k-t+1}^{k} p_\beta} (h^{\otimes}_{[1,r]} \otimes h^{\otimes}_{[r+1,r+k-i+1]}\mathfrak{M}_{k-i+1} \otimes h^{\otimes}_{[k-t+1,k-i+a']})\tau_{a'} \otimes h^{\otimes}_{[k-b+1,k]}\tau_b \\
&\quad + \sum_{i=2}^{k} \sum_{\substack{r+t=i-1\\ r,t\geq 0}} \sum_{\substack{a+b'=i\\ a\geq 1,\, b'\geq t+1}} \\
&\quad (-1)^{\sum_{\beta=k-t+1}^{k} p_\beta} h^{\otimes}_{[1,a]}\tau_a \otimes (h^{\otimes}_{[a+1,r]} \otimes h^{\otimes}_{[r+1,r+k-i+1]}\mathfrak{M}_{k-i+1} \otimes h^{\otimes}_{[k-t+1,k]})\tau_{b'} \\
&\quad + \Big(\sum_{i=2}^{k} \sum_{\substack{r+t=i-1\\ r,t\geq 0}} (-1)^{\sum_{\beta=k-t+1}^{k} p_\beta} \Big((h^{\otimes}_{[1,r]} \otimes h^{\otimes}_{[r+1,r+k-i+1]}\mathfrak{M}_{k-i+1} \otimes h^{\otimes}_{[k-t+1,k]})\tau_i\Big)\Big) \otimes f_k \Bigg)
\end{aligned} \tag{$****$}$$

We consider the second and third summand of $(****)$ separately. For the second one, we obtain

$$\sum_{i=2}^{k} \sum_{\substack{r+t=i-1\\ r,t\geq 0}} \sum_{\substack{a'+b=i\\ a'\geq r+1;\, b\geq 1}} (-1)^{\sum_{\beta=k-t+1}^{k} p_\beta} (h^{\otimes}_{[1,r]} \otimes h^{\otimes}_{[r+1,r+k-i+1]}\mathfrak{M}_{k-i+1} \otimes h^{\otimes}_{[k-t+1,k-i+a']})\tau_{a'} \otimes h^{\otimes}_{[k-b+1,k]}\tau_b$$

$$= \sum_{i=2}^{k} \sum_{\substack{a'+b=i \\ a',b\geq 1}} \sum_{\substack{r+t=i-1 \\ a'-1\geq r\geq 0;\, t\geq b}}$$

$$(-1)^{\sum_{\beta=k-t+1}^{k} p_\beta} \big(h^{\otimes}_{[1,r]} \otimes h^{\otimes}_{[r+1,r+k-i+1]} \mathfrak{M}_{k-i+1} \otimes h^{\otimes}_{[k-t+1,k-i+a']}\big)\tau_{a'} \otimes h^{\otimes}_{[k-b+1,k]}\tau_b$$

$$= \sum_{i=2}^{k} \sum_{\substack{a'+b=i \\ a',b\geq 1}} \sum_{\substack{r+u=a'-1 \\ r,u\geq 0}}$$

$$(-1)^{\sum_{\beta=k-b-u+1}^{k} p_\beta} \big(h^{\otimes}_{[1,r]} \otimes h^{\otimes}_{[r+1,r+k-i+1]} \mathfrak{M}_{k-i+1} \otimes h^{\otimes}_{[k-b-u+1,k-i+a']}\big)\tau_{a'} \otimes h^{\otimes}_{[k-b+1,k]}\tau_b$$

$$= \sum_{\substack{a'+b+j=k \\ a',b\geq 1;\, j\geq 0}} \sum_{\substack{r+u=a'-1 \\ r,u\geq 0}}$$

$$(-1)^{\sum_{\beta=k-b-u+1}^{k} p_\beta} \big(h^{\otimes}_{[1,r]} \otimes h^{\otimes}_{[r+1,r+j+1]} \mathfrak{M}_{j+1} \otimes h^{\otimes}_{[k-b-u+1,j+a']}\big)\tau_{a'} \otimes h^{\otimes}_{[k-b+1,k]}\tau_b$$

$$= \sum_{\substack{a+b=k \\ a,b\geq 1}} \sum_{a'=1}^{a} \sum_{\substack{r+u=a'-1 \\ r,u\geq 0}}$$

$$(-1)^{\sum_{\beta=k-b-u+1}^{k} p_\beta} \big(h^{\otimes}_{[1,r]} \otimes h^{\otimes}_{[r+1,r+a-a'+1]} \mathfrak{M}_{a-a'+1} \otimes h^{\otimes}_{[k-b-u+1,a]}\big)\tau_{a'} \otimes h^{\otimes}_{[k-b+1,k]}\tau_b$$

$$= \sum_{\substack{a+b=k \\ a,b\geq 1}} \sum_{i=1}^{a} \sum_{\substack{r+t=i-1 \\ r,t\geq 0}}$$

$$(-1)^{\sum_{\beta=a-t+1}^{k} p_\beta} \big(h^{\otimes}_{[1,r]} \otimes h^{\otimes}_{[r+1,r+a-i+1]} \mathfrak{M}_{a-i+1} \otimes h^{\otimes}_{[a-t+1,a]}\big)\tau_{i} \otimes h^{\otimes}_{[a+1,k]}\tau_b$$

$$= \sum_{\substack{a+b=k \\ a,b\geq 1}} \sum_{i=1}^{a} \sum_{\substack{r+t=i-1 \\ r,t\geq 0}} (-1)^{\sum_{\beta=a+1}^{k} p_\beta} \Big(h^{\otimes}_{[1,a]} \big(\mathrm{id}^{\otimes r} \otimes \mathfrak{M}_{a-i+1} \otimes \mathrm{id}^{\otimes t}\big)\Big)\tau_i \otimes h^{\otimes}_{[a+1,k]}\tau_b.$$

We proceed with the third summand of $(****)$.

$$\sum_{i=2}^{k} \sum_{\substack{r+t=i-1 \\ r,t\geq 0}} \sum_{\substack{a+b'=i \\ a\geq 1;\, b'\geq t+1}}$$

$$(-1)^{\sum_{\beta=k-t+1}^{k} p_\beta} h^{\otimes}_{[1,a]}\tau_a \otimes \big(h^{\otimes}_{[a+1,r]} \otimes h^{\otimes}_{[r+1,r+k-i+1]} \mathfrak{M}_{k-i+1} \otimes h^{\otimes}_{[k-t+1,k]}\big)\tau_{b'}$$

$$= \sum_{i=2}^{k} \sum_{\substack{a+b'=i \\ a,b'\geq 1}} \sum_{\substack{r+t=i-1 \\ r\geq a;\, b'-1\geq t\geq 0}}$$

$$(-1)^{\sum_{\beta=k-t+1}^{k} p_\beta} h^{\otimes}_{[1,a]}\tau_a \otimes \big(h^{\otimes}_{[a+1,r]} \otimes h^{\otimes}_{[r+1,r+k-i+1]} \mathfrak{M}_{k-i+1} \otimes h^{\otimes}_{[k-t+1,k]}\big)\tau_{b'}$$

$$= \sum_{i=2}^{k} \sum_{\substack{a+b'=i \\ a,b'\geq 1}} \sum_{\substack{u+t=b'-1 \\ u,t\geq 0}}$$

$$(-1)^{\sum_{\beta=k-t+1}^{k} p_\beta} h^{\otimes}_{[1,a]}\tau_a \otimes \big(h^{\otimes}_{[a+1,a+u]} \otimes h^{\otimes}_{[a+u+1,a+u+k-i+1]} \mathfrak{M}_{k-i+1} \otimes h^{\otimes}_{[k-t+1,k]}\big)\tau_{b'}$$

$$= \sum_{\substack{a+b'+j=k\\ a,b'\geq 1;\, j\geq 0}} \sum_{\substack{u+t=b'-1\\ u,t\geq 0}}$$

$$(-1)^{\sum_{\beta=k-t+1}^{k} p_\beta} h^{\otimes}_{[1,a]}\tau_a \otimes \big(h^{\otimes}_{[a+1,a+u]} \otimes h^{\otimes}_{[a+u+1,a+u+j+1]}\mathfrak{M}_{j+1} \otimes h^{\otimes}_{[k-t+1,k]}\big)\tau_{b'}$$

$$= \sum_{\substack{a+b=k\\ a,b\geq 1}} \sum_{b'=1}^{b} \sum_{\substack{u+t=b'-1\\ u,t\geq 0}}$$

$$(-1)^{\sum_{\beta=k-t+1}^{k} p_\beta} h^{\otimes}_{[1,a]}\tau_a \otimes \big(h^{\otimes}_{[a+1,a+u]} \otimes h^{\otimes}_{[a+u+1,a+u+b-b'+1]}\mathfrak{M}_{b-b'+1} \otimes h^{\otimes}_{[k-t+1,k]}\big)\tau_{b'}$$

$$= \sum_{\substack{a+b=k\\ a,b\geq 1}} \sum_{i=1}^{b} \sum_{\substack{r+t=i-1\\ r,t\geq 0}}$$

$$(-1)^{\sum_{\beta=k-t+1}^{k} p_\beta} h^{\otimes}_{[1,a]}\tau_a \otimes \big(h^{\otimes}_{[a+1,a+r]} \otimes h^{\otimes}_{[a+r+1,a+r+b-i+1]}\mathfrak{M}_{b-i+1} \otimes h^{\otimes}_{[k-t+1,k]}\big)\tau_i$$

$$= \sum_{\substack{a+b=k\\ a,b\geq 1}} \sum_{i=1}^{b} \sum_{\substack{r+t=i-1\\ r,t\geq 0}} h^{\otimes}_{[1,a]}\tau_a \otimes \Big(h^{\otimes}_{[a+1,k]}\big(\mathrm{id}^{\otimes r} \otimes \mathfrak{M}_{b-i+1} \otimes \mathrm{id}^{\otimes t}\big)\Big)\tau_i$$

With these two results, we go back to $(****)$ and obtain using the inductive hypothesis (IH), i.e. $(*_\ell)$ for $\ell < k$,

$$\sum_{i=2}^{k} \sum_{\substack{r+t=i-1\\ r,t\geq 0}} (-1)^{\sum_{\beta=k-t+1}^{k} p_\beta} \Big(\big(h^{\otimes}_{[1,r]} \otimes h^{\otimes}_{[r+1,r+k-i+1]}\mathfrak{M}_{k-i+1} \otimes h^{\otimes}_{[k-t+1,k]}\big)\mathfrak{t}_{i,1}\Big)\Delta$$

$$= \Delta\Bigg(\bigg(f_0 \otimes \Big(\sum_{i=2}^{k} \sum_{\substack{r+t=i-1\\ r,t\geq 0}} (-1)^{\sum_{\beta=k-t+1}^{k} p_\beta}\Big(\big(h^{\otimes}_{[1,r]} \otimes h^{\otimes}_{[r+1,r+k-i+1]}\mathfrak{M}_{k-i+1} \otimes h^{\otimes}_{[k-t+1,k]}\big)\mathfrak{t}_{i,1}\Big)\Big)$$

$$+ \sum_{\substack{a+b=k\\ a,b\geq 1}} \sum_{i=1}^{a} \sum_{\substack{r+t=i-1\\ r,t\geq 0}} (-1)^{\sum_{\beta=a+1}^{k} p_\beta} h^{\otimes}_{[1,a]}\big(\mathrm{id}^{\otimes r} \otimes \mathfrak{M}_{a-i+1} \otimes \mathrm{id}^{\otimes t}\big)\mathfrak{t}_{i,1} \otimes h^{\otimes}_{[a+1,k]}\mathfrak{t}_{b,1}$$

$$+ \sum_{\substack{a+b=k\\ a,b\geq 1}} \sum_{i=1}^{b} \sum_{\substack{r+t=i-1\\ r,t\geq 0}} h^{\otimes}_{[1,a]}\mathfrak{t}_{a,1} \otimes h^{\otimes}_{[a+1,k]}\big(\mathrm{id}^{\otimes r} \otimes \mathfrak{M}_{b-i+1} \otimes \mathrm{id}^{\otimes t}\big)\mathfrak{t}_{i,1}$$

$$+ \Big(\sum_{i=2}^{k} \sum_{\substack{r+t=i-1\\ r,t\geq 0}} (-1)^{\sum_{\beta=k-t+1}^{k} p_\beta}\Big(\big(h^{\otimes}_{[1,r]} \otimes h^{\otimes}_{[r+1,r+k-i+1]}\mathfrak{M}_{k-i+1} \otimes h^{\otimes}_{[k-t+1,k]}\big)\mathfrak{t}_{i,1}\Big)\Big) \otimes f_k\Bigg)$$

$$\overset{\text{(IH)}}{=} \Delta\Bigg(\bigg(f_0 \otimes \Big(\sum_{i=2}^{k} \sum_{\substack{r+t=i-1\\ r,t\geq 0}} \Big(h^{\otimes}_{[1,k]}\big(\mathrm{id}^{\otimes r} \otimes \mathfrak{M}_{k-i+1} \otimes \mathrm{id}^{\otimes t}\big)\mathfrak{t}_{i,1}\Big)\Big)$$

$$+ \sum_{\substack{a+b=k\\ a,b\geq 1}} (-1)^{\sum_{\beta=k-b+1}^{k} p_\beta} h^{\otimes}_{[1,a]}\mathfrak{t}_{a,1}\mathfrak{m}_{1,1} \otimes h^{\otimes}_{[k-b+1,k]}\mathfrak{t}_{b,1}$$

$$+ \sum_{\substack{a+b=k\\ a,b\geq 1}} h^{\otimes}_{[1,a]}\mathfrak{t}_{a,1} \otimes h^{\otimes}_{[k-b+1,k]}\mathfrak{t}_{b,1}\mathfrak{m}_{1,1}$$

$$+\Big(\sum_{i=2}^{k}\sum_{\substack{r+t=i-1\\ r,t\geq 0}}\Big(h^{\otimes}_{[1,k]}(\mathrm{id}^{\otimes r}\otimes\mathfrak{M}_{k-i+1}\otimes\mathrm{id}^{\otimes t})\mathfrak{t}_{i,1}\Big)\Big)\otimes f_k\Big)$$

Plugging in the previous result and the result of $(***)$ into $(**)$ we obtain

$$\begin{aligned}
&((h_1\otimes\ldots\otimes h_k)\tilde{\mathfrak{M}}_k)\Delta\\
&=((h_1\otimes\ldots\otimes h_k)\mathfrak{t}_{k,1}\mathfrak{m}_{1,1})\Delta\\
&-\sum_{i=2}^{k}\sum_{\substack{r+t=i-1\\ r,t\geq 0}}(-1)^{\sum_{\beta=k-t+1}^{k}p_\beta}\Big((h^{\otimes}_{[1,r]}\otimes h^{\otimes}_{[r+1,r+k-i+1]}\mathfrak{M}_{k-i+1}\otimes h^{\otimes}_{[k-t+1,k]})\mathfrak{t}_{i,1}\Big)\Delta\\
&=\Delta\Bigg(f_0\otimes h^{\otimes}_{[1,k]}\mathfrak{t}_{k,1}\mathfrak{m}_{1,1}+\Big(\sum_{\substack{a+b=k\\ a,b\geq 1}}h^{\otimes}_{[1,a]}\mathfrak{t}_{a,1}\otimes h^{\otimes}_{[k-b+1,k]}\mathfrak{t}_{b,1}\mathfrak{m}_{1,1}\Big)\\
&\quad+\Big(\sum_{\substack{a+b=k\\ a,b\geq 1}}(-1)^{\sum_{\beta=k-b+1}^{k}p_\beta}h^{\otimes}_{[1,a]}\mathfrak{t}_{a,1}\mathfrak{m}_{1,1}\otimes h^{\otimes}_{[k-b+1,k]}\mathfrak{t}_{b,1}\Big)+h^{\otimes}_{[1,k]}\mathfrak{t}_{k,1}\mathfrak{m}_{1,1}\otimes f_k\Bigg)\\
&-\Delta\Bigg(\Big(f_0\otimes\Big(\sum_{i=2}^{k}\sum_{\substack{r+t=i-1\\ r,t\geq 0}}\Big(h^{\otimes}_{[1,k]}(\mathrm{id}^{\otimes r}\otimes\mathfrak{M}_{k-i+1}\otimes\mathrm{id}^{\otimes t})\mathfrak{t}_{i,1}\Big)\Big)\\
&\quad+\Big(\sum_{\substack{a+b=k\\ a,b\geq 1}}(-1)^{\sum_{\beta=k-b+1}^{k}p_\beta}h^{\otimes}_{[1,a]}\mathfrak{t}_{a,1}\mathfrak{m}_{1,1}\otimes h^{\otimes}_{[k-b+1,k]}\mathfrak{t}_{b,1}\Big)\\
&\quad+\Big(\sum_{\substack{a+b=k\\ a,b\geq 1}}h^{\otimes}_{[1,a]}\mathfrak{t}_{a,1}\otimes h^{\otimes}_{[k-b+1,k]}\mathfrak{t}_{b,1}\mathfrak{m}_{1,1}\Big)\\
&\quad+\Big(\sum_{i=2}^{k}\sum_{\substack{r+t=i-1\\ r,t\geq 0}}\Big(h^{\otimes}_{[1,k]}(\mathrm{id}^{\otimes r}\otimes\mathfrak{M}_{k-i+1}\otimes\mathrm{id}^{\otimes t})\mathfrak{t}_{i,1}\Big)\Big)\otimes f_k\Bigg)\\
&=\Delta\big(f_0\otimes(h_1\otimes\ldots\otimes h_k)\tilde{\mathfrak{M}}_k+(h_1\otimes\ldots\otimes h_k)\tilde{\mathfrak{M}}_k\otimes f_k\big)
\end{aligned}$$

Hence the graded linear map $(h_1\otimes\ldots\otimes h_k)\tilde{\mathfrak{M}}_k$ is indeed an (f_0,f_k)-coderivation of degree $p+1$. So there is a $\mathbb{Z}_{TA,TB}$-graded linear map $\mathfrak{M}_k\colon\ \mathrm{Coder}(TA,TB)^{\otimes k}\to\mathrm{Coder}(TA,TB)$ of degree 1 such that $\tilde{\mathfrak{M}}_k=\mathfrak{M}_k\mathfrak{t}_{1,1}$. But then

$$\begin{aligned}
&\mathfrak{t}_{k,1}\mathfrak{m}_{1,1}-\sum_{i=1}^{k}\sum_{\substack{r+t=i-1\\ r,t\geq 0}}(\mathrm{id}^{\otimes r}\otimes\mathfrak{M}_{k-i+1}\otimes\mathrm{id}^{\otimes t})\mathfrak{t}_{i,1}\\
&\qquad=\mathfrak{t}_{k,1}\mathfrak{m}_{1,1}-\mathfrak{M}_k\mathfrak{t}_{1,1}-\sum_{i=2}^{k}\sum_{\substack{r+t=i-1\\ r,t\geq 0}}(\mathrm{id}^{\otimes r}\otimes\mathfrak{M}_{k-i+1}\otimes\mathrm{id}^{\otimes t})\mathfrak{t}_{i,1}\\
&\qquad=\mathfrak{t}_{k,1}\mathfrak{m}_{1,1}-\tilde{\mathfrak{M}}_k-\sum_{i=2}^{k}\sum_{\substack{r+t=i-1\\ r,t\geq 0}}(\mathrm{id}^{\otimes r}\otimes\mathfrak{M}_{k-i+1}\otimes\mathrm{id}^{\otimes t})\mathfrak{t}_{i,1}\\
&\qquad=0.
\end{aligned}$$

Hence we have constructed $\mathfrak{M}_k$ satisfying $(*_k)$. This proves the *claim*.

By Lemma 22.(2) the tuple $(\mathfrak{M}_k)_{k\geq 1}$ defines a $\mathcal{Z}_{TA,TB}$-graded $(\mathrm{id},\mathrm{id})$-coderivation

$$M\colon \quad T\,\mathrm{Coder}(TA,TB) \longrightarrow T\,\mathrm{Coder}(TA,TB)$$

of degree 1 with $M_{k,1} = \mathfrak{M}_k$ for $k \geq 1$. It remains to verify that $M\mathfrak{t} = \mathfrak{t}\mathfrak{m}$ and $M^2 = 0$.

By Lemma 36 the morphism $\mathfrak{t}\mathfrak{m} - M\mathfrak{t}$ is a $\mathcal{Z}_{TA,TB}$-graded $(\mathfrak{t},\mathfrak{t})$-coderivation of degree 1. Since both $\mathfrak{t}_{k,\ell} = 0$ and $M_{k,\ell} = 0$ for $k > \ell$ we have using Lemma 23 for $k \geq 1$

$$(\mathfrak{t}\mathfrak{m} - M\mathfrak{t})_{k,1} = \sum_{i=1}^{k} \mathfrak{t}_{k,i}\mathfrak{m}_{i,1} - \sum_{i=1}^{k} M_{k,i}\mathfrak{t}_{i,1}$$

But by Lemma 42 we have $\mathfrak{m}_{k,1} = 0$ for $k \geq 2$. Hence we obtain using Lemma 22.(2)

$$\begin{aligned}(\mathfrak{t}\mathfrak{m} - M\mathfrak{t})_{k,1} &= \mathfrak{t}_{k,1}\mathfrak{m}_{1,1} - \sum_{i=1}^{k} \sum_{\substack{r+s+t=k\\ r+1+t=i\\ r,t\geq 0,\, s\geq 1}} (\mathrm{id}^{\otimes r}\otimes M_{s,1}\otimes \mathrm{id}^{\otimes t})\mathfrak{t}_{i,1}\\ &= \mathfrak{t}_{k,1}\mathfrak{m}_{1,1} - \sum_{i=1}^{k} \sum_{\substack{r+s+t=k\\ r+1+t=i\\ r,t\geq 0,\, s\geq 1}} (\mathrm{id}^{\otimes r}\otimes \mathfrak{M}_{s}\otimes \mathrm{id}^{\otimes t})\mathfrak{t}_{i,1}\\ &\overset{(*_k)}{=} 0.\end{aligned}$$

Using Lemma 37 we conclude that $M\mathfrak{t} = \mathfrak{t}\mathfrak{m}$.

Finally, since $\mathfrak{m}^2 = 0$ we have $M^2\mathfrak{t} = M\mathfrak{t}\mathfrak{m} = \mathfrak{t}\mathfrak{m}^2 = 0$. But since $\mathfrak{t}$ is injective (cf. Lemma 48) it follows that $M^2 = 0$.

For the two formulas asserted in the end, we use again that $\mathfrak{t}_{k,\ell} = 0$ and $M_{k,\ell} = 0$ for $k < \ell$, cf. Lemma 23. Hence

$$0 = (\mathfrak{t}\mathfrak{m} - M\mathfrak{t})_{1,1} = \mathfrak{t}_{1,1}\mathfrak{m}_{1,1} - M_{1,1}\mathfrak{t}_{1,1}$$

and thus $M_{1,1}\mathfrak{t}_{1,1} = \mathfrak{t}_{1,1}\mathfrak{m}_{1,1}$. Secondly, we have

$$0 = (\mathfrak{t}\mathfrak{m} - M\mathfrak{t})_{2,1} = \mathfrak{t}_{2,2}\mathfrak{m}_{2,1} + \mathfrak{t}_{2,1}\mathfrak{m}_{1,1} - M_{2,2}\mathfrak{t}_{2,1} - M_{2,1}\mathfrak{t}_{1,1}.$$

But by Lemma 42 we have $\mathfrak{m}_{2,1} = 0$ and we have $M_{2,2} = \mathrm{id}\otimes M_{1,1} + M_{1,1}\otimes \mathrm{id}$ using Lemma 22.(2). Thus $M_{2,1}\mathfrak{t}_{1,1} = \mathfrak{t}_{2,1}\mathfrak{m}_{1,1} - (\mathrm{id}\otimes M_{1,1} + M_{1,1}\otimes \mathrm{id})\mathfrak{t}_{2,1}$. □

Remark 50 The differential M on $T\,\mathrm{Coder}(TA,TB)$ defines an A_∞-structure on the $\mathcal{Z}_{TA,TB}$-graded module of coderivations $\mathrm{Coder}(TA,TB)$. Since $\mathcal{Z}_{TA,TB} = \mathbf{Z}\times \mathsf{Pair}(\mathsf{dgCoalg}(TA,TB))$, this A_∞-structure is actually an A_∞-category with the set of differential graded coalgebra morphisms as objects.

This A_∞-structure has already been constructed by Fukaya [Fuk02], Lyubashenko [Lyu03] and Lefèvre-Hasegawa [Lef03]. Our approach given here is similar to the one presented in [Lyu03] by Lyubashenko, in the sense that Lyubashenko also works on the differential graded coalgebra side of the bar construction and not on the A_∞-algebra side.

Lemma 51 *Suppose given* $f_0, f_1, f_2 \in \mathsf{dgCoalg}(TA, TB)$.
Suppose given an (f_0, f_1)*-coderivation* $h_1\colon TA \to TB$ *of degree* p_1 *and an* (f_1, f_2)*-coderivation* $h_2\colon TA \to TB$ *of degree* p_2. *Then the following equality of graded linear maps from* $A^{\otimes k}$ *to* B *holds for* $k \geq 1$.

$$\begin{aligned}&\big((h_1 \otimes h_2)M_{2,1}^{p_1+p_2,(f_0,f_2)}\big)_{k,1}\\&\quad= \sum_{\substack{r_0+s_1+r_1+s_2+r_2=k\\ r_0,r_1,r_2,r_0',r_1',r_2'\geq 0,\, s_1,s_2\geq 1}} \Big((\hat{f}_0)_{r_0,r_0'} \otimes (h_1)_{s_1,1} \otimes (\hat{f}_1)_{r_1,r_1'} \otimes (h_2)_{s_2,1} \otimes (\hat{f}_2)_{r_2,r_2'}\Big) m_{r_0'+1+r_1'+1+r_2',1}\end{aligned}$$

Proof. Since $\mathfrak{t}_{1,1}^{p_1+p_2,(f_0,f_2)} = \tau_1$ by Definition 47 and since by Remark 45.(1) the morphism $\tau_1\colon \operatorname{Coder}(TA,TB)^{p_1+p_2,(f_0,f_2)} \to \mathsf{grHom}(TA,TB)^{p_1+p_2}$ is the inclusion we have

$$(h_1 \otimes h_2)M_{2,1}^{p_1+p_2,(f_0,f_2)} = (h_1 \otimes h_2)M_{2,1}^{p_1+p_2,(f_0,f_2)}\mathfrak{t}_{1,1}^{p_1+p_2,(f_0,f_2)}.$$

Theorem 49 with Lemma 42 and Definition 47 then gives

$$\begin{aligned}(h_1 \otimes h_2)M_{2,1}^{p_1+p_2,(f_0,f_2)} &= (h_1 \otimes h_2)M_{2,1}^{p_1+p_2,(f_0,f_2)}\mathfrak{t}_{1,1}^{p_1+p_2,(f_0,f_2)}\\ &= (h_1 \otimes h_2)\mathfrak{t}_{2,1}^{p_1+p_2,(f_0,f_2)}\mathfrak{m}_{1,1}^{p_1+p_2,(f_0,f_2)}\\ &\quad - (h_1 \otimes h_2 M_{1,1}^{p_2,(f_1,f_2)})\mathfrak{t}_{2,1}^{p_1+p_2+1,(f_0,f_2)}\\ &\qquad - (-1)^{p_2}(h_1 M_{1,1}^{p_1,(f_0,f_1)} \otimes h_2)\mathfrak{t}_{2,1}^{p_1+p_2+1,(f_0,f_2)}\\ &= ((h_1 \otimes h_2)\tau_2\mu^{p_1+p_2}\\ &\quad - (h_1 \otimes h_2 M_{1,1}^{p_2,(f_1,f_2)})\tau_2 - (-1)^{p_2}(h_1 M_{1,1}^{p_1,(f_0,f_1)} \otimes h_2)\tau_2.\end{aligned}$$

Note that by Remark 44.(2) we have $\big((h_1 \otimes h_2)\tau_2\big)_{k,1} = 0$ for $k \geq 1$ and arbitrary coderivations h_1 and h_2. Thus using Lemma 40.(1)

$$\begin{aligned}\big((h_1 \otimes h_2)M_{2,1}^{p_1+p_2,(f_0,f_2)}\big)_{k,1} &= \big((h_1 \otimes h_2)\tau_2\mu^{p_1+p_2}\big)_{k,1}\\ &= \big(((h_1 \otimes h_2)\tau_2)m - (-1)^{p_1+p_2}m((h_1 \otimes h_2)\tau_2)\big)_{k,1}\\ &= \big(((h_1 \otimes h_2)\tau_2)m\big)_{k,1}\end{aligned}$$

We obtain using Definition 43 and Remark 44.(1)

$$\begin{aligned}&\big(((h_1 \otimes h_2)\tau_2)m\big)_{k,1}\\ &\quad= \sum_{\ell=1}^{k}((h_1 \otimes h_2)\tau_2)_{k,\ell}m_{\ell,1}\\ &\quad= \sum_{\ell=1}^{k}\sum_{\substack{r_0+s_1+r_1+s_2+r_2=k\\ r_0'+1+r_1'+1+r_2'=\ell\\ r_0,r_1,r_2,r_0',r_1',r_2'\geq 0,\, s_1,s_2\geq 1}} \Big((\hat{f}_0)_{r_0,r_0'} \otimes (h_1)_{s_1,1} \otimes (\hat{f}_1)_{r_1,r_1'} \otimes (h_2)_{s_2,1} \otimes (\hat{f}_2)_{r_2,r_2'}\Big) m_{\ell,1}\\ &\quad= \sum_{\substack{r_0+s_1+r_1+s_2+r_2=k\\ r_0,r_1,r_2,r_0',r_1',r_2'\geq 0,\, s_1,s_2\geq 1}} \Big((\hat{f}_0)_{r_0,r_0'} \otimes (h_1)_{s_1,1} \otimes (\hat{f}_1)_{r_1,r_1'} \otimes (h_2)_{s_2,1} \otimes (\hat{f}_2)_{r_2,r_2'}\Big) m_{r_0'+1+r_1'+1+r_2',1}\end{aligned}$$

□

2.2 Homotopies

Let A and B be graded modules.

Suppose we are given differential graded tensor coalgebras (TA, Δ, m) and (TB, Δ, m), cf. Definition 29.

In this section we prove that coderivation homotopy, cf. Definition 57 below, is an equivalence relation on the set of differential graded coalgebra morphisms from TA to TB, cf. Lemma 61. To prove e.g. symmetry, we need to turn (f,g)-coderivations into (g,f)-coderivations. For this, we introduce and study the transfer morphism in §2.2.1.

2.2.1 Transferring coderivations

Suppose given morphisms of differential graded coalgebras $f\colon TA \to TB$ and $g\colon TA \to TB$

We write $\mathrm{Coder}(TA, TB)^{(f,g)}$ for the $\mathbf{Z}$-graded module that has at $p \in \mathbf{Z}$ the module $\mathrm{Coder}(TA, TB)^{p,(f,g)}$ of (f,g)-coderivations of degree p.

By Lemma 37 there is an isomorphism of $\mathbf{Z}$-graded modules of degree 0

$$\begin{array}{rrcl} \beta_{f,g}\colon & \mathrm{Coder}(TA,TB)^{(f,g)} & \longrightarrow & \mathsf{grHom}(TA,B) \\ \beta_{f,g}^{p}\colon & h & \longmapsto & h\pi_1 . \end{array}$$

Definition 52 Suppose given $f_1, f_2, g_1, g_2 \in \mathsf{dgCoalg}(TA, TB)$.

The *transfer isomorphism* from $\mathrm{Coder}(TA, TB)^{(f_1,g_1)}$ to $\mathrm{Coder}(TA, TB)^{(f_2,g_2)}$ is the isomorphism of $\mathbf{Z}$-graded modules of degree 0

$$\Phi_{f_1,g_1}^{f_2,g_2}\colon \quad \mathrm{Coder}(TA,TB)^{(f_1,g_1)} \longrightarrow \mathrm{Coder}(TA,TB)^{(f_2,g_2)}$$

given by $\Phi_{f_1,g_1}^{f_2,g_2} := \beta_{f_1,g_1}(\beta_{f_2,g_2})^{-1}$.

Recall that we often write $\Phi_{f_1,g_1}^{f_2,g_2} := (\Phi_{f_1,g_1}^{f_2,g_2})^p$ for $p \in \mathbf{Z}$.

Lemma 53 *Suppose given* $f_1, f_2, g_1, g_2 \in \mathsf{dgCoalg}(TA, TB)$.

Then the following formula holds for an (f_1, g_1)*-coderivation* $h\colon TA \to TB$ *of degree* $p \in \mathbf{Z}$.

$$h\Phi_{f_1,g_1}^{f_2,g_2} = h + ((f_2 - f_1) \otimes h)\tau_2 - (h \otimes (g_1 - g_2))\tau_2 - ((f_2 - f_1) \otimes h \otimes (g_1 - g_2))\tau_3$$

For the graded linear maps τ_2 *and* τ_3 *see Definition* 43.

Proof. We show that the right-hand side is an (f_2, g_2)-coderivation of degree p. We calculate using Lemma 46.

$$\begin{aligned}
&(h+((f_2 - f_1) \otimes h)\tau_2 - (h \otimes (g_1 - g_2))\tau_2 - ((f_2 - f_1) \otimes h \otimes (g_1 - g_2))\tau_3)\Delta \\
&\quad = \Delta\Big(f_1 \otimes h + h \otimes g_1 \\
&\qquad + f_2 \otimes ((f_2 - f_1) \otimes h)\tau_2 + (f_2 - f_1) \otimes h + ((f_2 - f_1) \otimes h)\tau_2 \otimes g_1 \\
&\qquad - f_1 \otimes (h \otimes (g_1 - g_2))\tau_2 - h \otimes (g_1 - g_2) - (h \otimes (g_1 - g_2))\tau_2 \otimes g_2 \\
&\qquad - f_2 \otimes ((f_2 - f_1) \otimes h \otimes (g_1 - g_2))\tau_3 - (f_2 - f_1) \otimes (h \otimes (g_1 - g_2))\tau_2 \\
&\qquad - ((f_2 - f_1) \otimes h)\tau_2 \otimes (g_1 - g_2) - ((f_2 - f_1) \otimes h \otimes (g_1 - g_2))\tau_3) \otimes g_2\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \Delta\Big(f_2 \otimes h + h \otimes g_2 \\
&\quad + f_2 \otimes \big((f_2 - f_1) \otimes h\big)\tau_2 + \big((f_2 - f_1) \otimes h\big)\tau_2 \otimes g_2 \\
&\quad - f_2 \otimes \big(h \otimes (g_1 - g_2)\big)\tau_2 - \big(h \otimes (g_1 - g_2)\big)\tau_2 \otimes g_2 \\
&\quad - f_2 \otimes \big((f_2 - f_1) \otimes h \otimes (g_1 - g_2)\big)\tau_3 \\
&\quad - \big((f_2 - f_1) \otimes h \otimes (g_1 - g_2)\big)\tau_3 \otimes g_2\Big) \\
&= \Delta\Big(f_2 \otimes \big((h + h \otimes (g_1 - g_2)\big)\tau_2 - \big((f_2 - f_1) \otimes h\big)\tau_2 - \big((f_2 - f_1) \otimes h \otimes (g_1 - g_2)\big)\tau_3\big) \\
&\quad + \big(h + (h \otimes (g_1 - g_2)\big)\tau_2 - \big((f_2 - f_1) \otimes h\big)\tau_2 - \big((f_2 - f_1) \otimes h \otimes (g_1 - g_2)\big)\tau_3\big) \otimes g_2\Big)
\end{aligned}
$$

Hence the right-hand side is an (f_2, g_2)-coderivation, so we can apply the isomorphism β_{f_2,g_2} to it.

$$
\begin{aligned}
&(h + ((f_2 - f_1) \otimes h)\tau_2 - (h \otimes (g_1 - g_2))\tau_2 - ((f_2 - f_1) \otimes h \otimes (g_1 - g_2))\tau_3)\beta_{f_2,g_2} \\
&\quad = h\pi_1 + ((f_2 - f_1) \otimes h)\tau_2\pi_1 - (h \otimes (g_1 - g_2))\tau_2\pi_1 - ((f_2 - f_1) \otimes h \otimes (g_1 - g_2))\tau_3\pi_1 \\
&\quad = h\pi_1 \\
&\quad = h\beta_{f_1,g_1}.
\end{aligned}
$$

Here we used that for $n \geq 2$ one has $\big((h_1 \otimes \ldots \otimes h_n)\tau_n\big)_{k,1} = 0$ for $k \geq 2$, cf. Remark 44.(2). The assertion follows now by applying $(\beta_{f_2,g_2})^{-1}$ to the above equation. □

Lemma 54 *Suppose given $f_0, f_1, f_2 \in \mathsf{dgCoalg}(TA, TB)$. Then the following holds.*

$$(f_0 - f_1)\Phi^{f_0,f_2}_{f_0,f_1} + (f_1 - f_2)\Phi^{f_0,f_2}_{f_1,f_2} = f_0 - f_2$$

Proof. After application of β_{f_0,f_2} we have to show that

$$(f_0 - f_1)\beta_{f_0,f_1} + (f_1 - f_2)\beta_{f_1,f_2} = (f_0 - f_2)\beta_{f_0,f_2},$$

cf. Definition 52. But we have

$$(f_0 - f_1)\pi_1 + (f_1 - f_2)\pi_1 = (f_0 - f_2)\pi_1,$$

hence the assertion follows. □

Remark 55 Suppose given morphisms of differential graded coalgebras $f, g \in \mathsf{dgCoalg}(TA, TB)$ and an (f, g)-coderivation $h\colon TA \to TB$ of degree p.
Recall that $\mathfrak{t}^{p,(f,g)}_{1,1} = \tau_1\colon \operatorname{Coder}(TA, TB)^{p,(f,g)} \to \operatorname{PreCoder}(TA, TB)^{p,(f,g)} = \mathsf{grHom}(TA, TB)^p$ is the inclusion, i.e. we have $h\mathfrak{t}^{p,(f,g)}_{1,1} = h$, cf. Remark 45.(1).
By Theorem 49 we have $M_{1,1}\mathfrak{t}_{1,1} = \mathfrak{t}_{1,1}\mathfrak{m}_{1,1}$. With Lemma 42 it follows that

$$hM^{p,(f,g)}_{1,1} = h\mathfrak{m}^{p,(f,g)}_{1,1} = h\mu^p$$

with the differential μ from Lemma 40.

Lemma 56 *Suppose given $f_1, f_2, g_1, g_2 \in \mathsf{dgCoalg}(TA, TB)$.*
For an (f_1, g_1)-coderivation $h\colon TA \to TB$ of degree p the following hold.

(1) $h\Phi^{f_1,g_2}_{f_1,g_1} M^{p,(f_1,g_2)}_{1,1} - hM^{p,(f_1,g_1)}_{1,1}\Phi^{f_1,g_2}_{f_1,g_1} = -(h \otimes (g_1 - g_2))M^{p,(f_1,g_2)}_{2,1}$

(2) $h\Phi^{f_2,g_1}_{f_1,g_1} M^{p,(f_2,g_1)}_{1,1} - hM^{p,(f_1,g_1)}_{1,1}\Phi^{f_2,g_1}_{f_1,g_1} = ((f_2 - f_1) \otimes h)M^{p,(f_2,g_1)}_{2,1}$

Proof. Recall the $\mathcal{Z}_{TA,TB}$-graded coalgebra morphism

$$\mathfrak{t}\colon \quad T\,\mathrm{Coder}(TA,TB) \longrightarrow T\,\mathrm{PreCoder}(TA,TB)$$

with $\mathfrak{t}^{p,(f,g)}_{k,1} = \tau_k$ with the τ_k from Definition 43 for $k \geq 1$, $p \in \mathbf{Z}$ and $f, g \in \mathsf{dgCoalg}(TA,TB)$, cf. Definition 47.
By Theorem 49 the following formula holds.

$$M_{2,1}\mathfrak{t}_{1,1} = \mathfrak{t}_{2,1}\mathfrak{m}_{1,1} - (\mathrm{id} \otimes M_{1,1} + M_{1,1} \otimes \mathrm{id})\mathfrak{t}_{2,1}.$$

Given $\varphi_0, \varphi_1, \varphi_2 \in \mathsf{dgCoalg}(TA,TB)$ and an (φ_0,φ_1)-coderivation $\eta_1\colon TA \to TB$ of degree p_1 and an (φ_1,φ_2)-coderivation $\eta_2\colon TA \to TB$ of degree p_2 this implies with Remark 55 that as graded linear maps we have

$$(\eta_1 \otimes \eta_2)M^{p_1+p_2,(\varphi_0,\varphi_2)}_{2,1} = (\eta_1 \otimes \eta_2)\tau_2\mu^{p_1+p_2} - (\eta_1 \otimes \eta_2\mu^{p_2} + (-1)^{p_2}\eta_1\mu^{p_1} \otimes \eta_2)\tau_2. \qquad (*)$$

Moreover, note that $(\varphi_1 - \varphi_0)\mu^0 = m(\varphi_1 - \varphi_0) - (\varphi_1 - \varphi_0)m = 0$.
Suppose given an (f_1, g_1)-coderivation $h\colon TA \to TB$ of degree p.
For (1), we calculate using Lemma 53.

$$\begin{aligned}
&h\Phi^{f_1,g_2}_{f_1,g_1} M^{p,(f_1,g_2)}_{1,1} - hM^{p,(f_1,g_1)}_{1,1}\Phi^{f_1,g_2}_{f_1,g_1}\\
&\overset{\text{L 53}}{=} \Big(h - (h \otimes (g_1 - g_2))\tau_2\Big)M^{p,(f_1,g_2)}_{1,1} - hM^{p,(f_1,g_1)}_{1,1} + \big(hM^{p,(f_1,g_1)}_{1,1} \otimes (g_1 - g_2)\big)\tau_2\\
&= \Big(h - (h \otimes (g_1 - g_2))\tau_2\Big)\mu^p - h\mu^p + \big(h\mu^p \otimes (g_1 - g_2)\big)\tau_2\\
&= -(h \otimes (g_1 - g_2))\tau_2\mu^p + \big(h \otimes (g_1 - g_2)\mu^0 + h\mu^p \otimes (g_1 - g_2)\big)\tau_2\\
&\overset{(*)}{=} -(h \otimes (g_1 - g_2))M^{p,(f_1,g_2)}_{2,1}
\end{aligned}$$

For (2), we also calculate using Lemma 53.

$$\begin{aligned}
&h\Phi^{f_2,g_1}_{f_1,g_1} M^{p,(f_2,g_1)}_{1,1} - hM^{p,(f_1,g_1)}_{1,1}\Phi^{f_2,g_1}_{f_1,g_1}\\
&\overset{\text{L 53}}{=} \Big(h + ((f_2 - f_1) \otimes h)\tau_2\Big)M^{p,(f_2,g_1)}_{1,1} - hM^{p,(f_1,g_1)}_{1,1} - \big((f_2 - f_1) \otimes hM^{p,(f_1,g_1)}_{1,1}\big)\tau_2\\
&= \Big(h + ((f_2 - f_1) \otimes h)\tau_2\Big)\mu^p - h\mu^p - \big((f_2 - f_1) \otimes h\mu^p\big)\tau_2\\
&= ((f_2 - f_1) \otimes h)\tau_2\mu^p - \big((f_2 - f_1) \otimes h\mu^p + (-1)^p(f_2 - f_1)\mu^0 \otimes h\big)\tau_2\\
&\overset{(*)}{=} ((f_2 - f_1) \otimes h)M^{p,(f_2,g_1)}_{2,1}
\end{aligned}$$

□

2.2.2 Coderivation homotopy

We are now in a position to define coderivation homotopy on differential graded tensor coalgebras and prove that it is an equivalence relation.

Definition 57 Let $f\colon TA \to TB$ and $g\colon TA \to TB$ be morphisms of differential graded coalgebras.

A *coderivation homotopy* from f to g is an (f,g)-coderivation $h\colon TA \to TB$ of degree -1 such that $f - g = hm + mh$, cf. Definition 34.

We call the morphisms f and g *coderivation homotopic* if there exists a coderivation homotopy from f to g.

We sometimes just write *homotopy* for *coderivation homotopy.*

Lemma 58 *Let A', A, B, B' be graded modules. Suppose we are given differential graded tensor coalgebras (TA', Δ, m), (TA, Δ, m), (TB, Δ, m) and (TB', Δ, m), i.e. objects in* $\mathsf{dtCoalg}$, *cf. Definition* 29.

Suppose given morphisms of differential graded coalgebras $f\colon TA \to TB$ and $g\colon TA \to TB$, $s\colon TA' \to TA$ and $t\colon TB \to TB'$. Suppose that $h\colon TA \to TB$ is a coderivation homotopy from f to g.

Then $sht\colon TA' \to TB'$ is a coderivation homotopy from sft to sgt.

Proof. By Lemma 36 the graded linear map $sht\colon TA' \to TB'$ is an (sft, sgt)-coderivation of degree -1. Moreover, we have

$$sft - sgt = s(f-g)t = s(hm + mh)t = shmt + smht = shtm + msht,$$

since s and t are morphisms of differential graded coalgebras and thus commute with the differentials. It follows that sht is a coderivation homotopy from sft to sgt. □

Remark 59 Let $f, g \in \mathsf{dgCoalg}(TA, TB)$ be morphisms of differential graded coalgebras.

By Remark 35 we know that $f - g$ is an (f,g)-coderivation of degree 0. Using Remark 55 and Lemma 40 we have for an (f,g)-coderivation $h\colon TA \to TB$ of degree p that

$$hM_{1,1}^{p,(f,g)} = h\mathfrak{m}_{1,1}^{p,(f,g)} = h\mu^p = hm - (-1)^p mh.$$

So h is a coderivation homotopy from f to g if and only if h is an (f,g)-coderivation of degree -1 and satisfies

$$hM_{1,1}^{-1,(f,g)} = f - g.$$

Recall the $\mathbf{Z}$-graded module $\operatorname{Coder}(TA,TB)^{(f,g)}$ of (f,g)-coderivations that has at $p \in \mathbf{Z}$ the module $\operatorname{Coder}(TA,TB)^{p,(f,g)}$ of (f,g)-coderivations of degree p. Then $\operatorname{Coder}(TA,TB)^{(f,g)}$ becomes a differential $\mathbf{Z}$-graded module (i.e. a complex) with the differential $M_{1,1}^{(f,g)}$ which is at $p \in \mathbf{Z}$ given by $\big(M_{1,1}^{(f,g)}\big)^p := M_{1,1}^{p,(f,g)}$.

Lemma 60 *Let $f, g \in \mathsf{dgCoalg}(TA, TB)$ be morphisms of differential graded coalgebras. Suppose there exists a coderivation homotopy $h'\colon TA \to TB$ from f to g. Consider the following $\mathbf{Z}$-graded linear maps of degree* 0.

$$\begin{array}{llcl}
\Psi_{h'\uparrow}\colon & \operatorname{Coder}(TA,TB)^{(g,f)} & \longrightarrow & \operatorname{Coder}(TA,TB)^{(g,g)} \\
\Psi_{h'\uparrow}^{p}\colon & h & \longmapsto & -h(\Phi_{g,f}^{g,g})^p + (h \otimes h')M_{2,1}^{p-1,(g,g)}
\end{array}$$

$$\begin{array}{llcl}
\Psi_{h'\downarrow}\colon & \operatorname{Coder}(TA,TB)^{(g,g)} & \longrightarrow & \operatorname{Coder}(TA,TB)^{(f,g)} \\
\Psi_{h'\downarrow}^{p}\colon & h & \longmapsto & h(\Phi_{g,g}^{f,g})^p + (-1)^p(h' \otimes h)M_{2,1}^{p-1,(f,g)}
\end{array}$$

Then $\Psi_{h'\upharpoonleft}$ and $\Psi_{h'\downharpoonleft}$ are isomorphisms of differential **Z***-graded modules.*

$$\begin{array}{ccccc} & & (\mathrm{Coder}(TA,TB)^{(g,g)}, M_{1,1}^{(g,g)}) & & \\ & \overset{\Psi_{h'\upharpoonleft}}{\underset{\sim}{\nearrow}} & & \overset{\Psi_{h'\downharpoonleft}}{\underset{\sim}{\searrow}} & \\ (\mathrm{Coder}(TA,TB)^{(g,f)}, M_{1,1}^{(g,f)}) & & & & (\mathrm{Coder}(TA,TB)^{(f,g)}, M_{1,1}^{(f,g)}) \end{array}$$

Proof. Since M is a differential on $T\,\mathrm{Coder}(TA,TB)$ by Theorem 49, the tuple $(M_{k,1})_{k\geq 1}$ satisfies the Stasheff equations by Lemma 24.(1). In particular, we have

$$M_{1,1}M_{1,1} = 0 \quad \text{and} \quad 0 = M_{2,1}M_{1,1} + (\mathrm{id}\otimes M_{1,1} + M_{1,1}\otimes \mathrm{id})M_{2,1}. \tag{$*$}$$

We first show that $\Psi_{h'\upharpoonleft}$ and $\Psi_{h'\downharpoonleft}$ are morphisms of differential **Z**-graded modules, i.e. we show that $\Psi_{h'\upharpoonleft}M_{1,1}^{(g,g)} = M_{1,1}^{(g,f)}\Psi_{h'\upharpoonleft}$ and $\Psi_{h'\downharpoonleft}M_{1,1}^{(f,g)} = M_{1,1}^{(g,g)}\Psi_{h'\downharpoonleft}$.
For $\Psi_{h'\upharpoonleft}$, let $h\colon TA\to TB$ be a (g,f)-coderivation of degree p. We obtain using $(*)$, Remark 59 and Lemma 56.(1)

$$\begin{aligned} h\Psi^p_{h'\upharpoonleft}M_{1,1}^{p,(g,g)} &= -h\Phi^{g,g}_{g,f}M_{1,1}^{p,(g,g)} + (h\otimes h')M_{2,1}^{p-1,(g,g)}M_{1,1}^{p,(g,g)} \\ &= -hM_{1,1}^{p,(g,f)}\Phi^{g,g}_{g,f} + \big(h\otimes (f-g)\big)M_{2,1}^{p,(g,g)} \\ &\quad - \big(h\otimes h'M_{1,1}^{-1,(f,g)}\big)M_{2,1}^{p,(g,g)} + \big(hM_{1,1}^{p,(g,f)}\otimes h'\big)M_{2,1}^{p,(g,g)} \\ &= -hM_{1,1}^{p,(g,f)}\Phi^{g,g}_{g,f} + \big(hM_{1,1}^{p,(g,f)}\otimes h'\big)M_{2,1}^{p,(g,g)} \\ &\quad + \big(h\otimes (f-g)\big)M_{2,1}^{p,(g,g)} - \big(h\otimes (f-g)\big)M_{2,1}^{p,(g,g)} \\ &= hM_{1,1}^{p,(g,f)}\Psi^{p+1}_{h'\upharpoonleft}. \end{aligned}$$

For $\Psi_{h'\downharpoonleft}$, let $h\colon TA\to TB$ be a (g,g)-coderivation of degree p. We obtain using $(*)$, Remark 59 and Lemma 56.(2)

$$\begin{aligned} h\Psi^p_{h'\downharpoonleft}M_{1,1}^{p,(f,g)} &= h\Phi^{f,g}_{g,g}M_{1,1}^{p,(f,g)} + (-1)^p(h'\otimes h)M_{2,1}^{p-1,(f,g)}M_{1,1}^{p,(f,g)} \\ &= hM_{1,1}^{p,(g,g)}\Phi^{f,g}_{g,g} + \big((f-g)\otimes h\big)M_{2,1}^{p,(f,g)} \\ &\quad - (-1)^p\big(h'\otimes hM_{1,1}^{p,(g,g)}\big)M_{2,1}^{p,(f,g)} - (-1)^p(-1)^p\big(h'M_{1,1}^{-1,(f,g)}\otimes h\big)M_{2,1}^{p,(f,g)} \\ &= hM_{1,1}^{p,(g,g)}\Phi^{f,g}_{g,g} + (-1)^{p+1}\big(h'\otimes hM_{1,1}^{p,(g,g)}\big)M_{2,1}^{p,(f,g)} \\ &\quad + \big((f-g)\otimes h\big)M_{2,1}^{p,(f,g)} - \big((f-g)\otimes h\big)M_{2,1}^{p,(f,g)} \\ &= hM_{1,1}^{p,(g,g)}\Psi^{p+1}_{h'\downharpoonleft}. \end{aligned}$$

It remains to show that $\Psi_{h'\upharpoonleft}$ and $\Psi_{h'\downharpoonleft}$ are isomorphisms of **Z**-graded modules. For $p\in\mathbf{Z}$, recall the isomorphisms $\beta^p_{f,g}$, $\beta^p_{g,g}$ and $\beta^p_{g,f}$ from Lemma 37, which are all given by $h\mapsto h\pi_1$. Define linear maps $\psi^p_{h'\upharpoonleft}$ and $\psi^p_{h'\downharpoonleft}$ such that the following diagram commutes.

$$\begin{array}{ccccc} \mathrm{Coder}(TA,TB)^{p,(g,f)} & \xrightarrow{\Psi^p_{h'\upharpoonleft}} & \mathrm{Coder}(TA,TB)^{p,(g,g)} & \xrightarrow{\Psi^p_{h'\downharpoonleft}} & \mathrm{Coder}(TA,TB)^{p,(f,g)} \\ \wr\downarrow \beta^p_{g,f} & & \wr\downarrow \beta^p_{g,g} & & \wr\downarrow \beta^p_{f,g} \\ \mathsf{grHom}(TA,B)^p & \xrightarrow{\psi^p_{h'\upharpoonleft}} & \mathsf{grHom}(TA,B)^p & \xrightarrow{\psi^p_{h'\downharpoonleft}} & \mathsf{grHom}(TA,B)^p \end{array}$$

It suffices to show that $\psi^p_{h'\upharpoonleft}$ and $\psi^p_{h'\downharpoonleft}$ are isomorphisms.

For $\psi^p_{h'\upharpoonleft}$, let $\eta\colon TA \to B$ be a graded linear map of degree p and let $h\colon TA \to TB$ be the unique (g,f)-coderivation of degree p such that $h\beta^p_{g,f} = \eta$. For $k \geq 1$ we have using Lemma 51

$$\begin{aligned}
(\eta\psi^p_{h'\upharpoonleft})_k &= \iota_k(\eta\psi^p_{h'\upharpoonleft}) \\
&= \iota_k(h\beta^p_{g,f}\psi^p_{h'\upharpoonleft}) \\
&= \iota_k(h\Psi^p_{h'\upharpoonleft}\beta^p_{g,g}) \\
&= \iota_k\big((-h\Phi^{g,g}_{g,f} + (h\otimes h')M^{p-1,(g,g)}_{2,1})\beta^p_{g,g}\big) \\
&= -\iota_k(h\beta^p_{g,f}) + \iota_k\big((h\otimes h')M^{p-1,(g,g)}_{2,1}\beta^p_{g,g}\big) \\
&= -\iota_k\eta + \iota_k\big((h\otimes h')M^{p-1,(g,g)}_{2,1}\big)\pi_1 \\
&= -\eta_k + \sum_{\substack{r_0+s_1+r_1+s_2+r_2=k\\ r_0,r_1,r_2,r'_0,r'_1,r'_2\geq 0\\ s_1,s_2\geq 1}} \big((\hat{g})_{r_0,r'_0}\otimes \eta_{s_1}\otimes (\hat{f})_{r_1,r'_1}\otimes (h')_{s_2,1}\otimes (\hat{g})_{r_2,r'_2}\big)m_{r'_0+1+r'_1+1+r'_2,1}.
\end{aligned}$$

Injectivity of $\psi^p_{h'\upharpoonleft}$. Suppose that $\eta\psi^p_{h'\upharpoonleft} = 0$, i.e. $(\eta\psi^p_{h'\upharpoonleft})_k = 0$ for $k \geq 1$. We show that $\eta_k = 0$ for $k \geq 1$ by induction on k. For $k = 1$ note that by the above formula $(\eta\psi^p_{h'\upharpoonleft})_1 = -\eta_1$, i.e. $\eta_1 = 0$. Now let $k > 1$ and suppose that $\eta_\ell = 0$ for $\ell < k$. But then the above formula for $(\eta\psi^p_{h'\upharpoonleft})_k$ implies that $(\eta\psi^p_{h'\upharpoonleft})_k = -\eta_k$, since in the sum only terms η_{s_1} with $s_1 < k$ appear. Thus $\eta_k = 0$. Hence $\ker(\psi^p_{h'\upharpoonleft}) = \{0\}$ and we conclude that $\psi^p_{h'\upharpoonleft}$ is injective.

Surjectivity of $\psi^p_{h'\upharpoonleft}$. Suppose given a graded linear map $\theta\colon TA \to B$ of degree p. We construct the components $\eta_k\colon A^{\otimes k} \to B$ of a graded linear map $\eta\colon TA \to B$ of degree p by the following recursive formula for $k \geq 1$.

$$\eta_k := -\theta_k + \sum_{\substack{r_0+s_1+r_1+s_2+r_2=k\\ r_0,r_1,r_2,r'_0,r'_1,r'_2\geq 0\\ s_1,s_2\geq 1}} \big((\hat{g})_{r_0,r'_0}\otimes \eta_{s_1}\otimes (\hat{f})_{r_1,r'_1}\otimes (h')_{s_2,1}\otimes (\hat{g})_{r_2,r'_2}\big)m_{r'_0+1+r'_1+1+r'_2,1}$$

Note that in the above sum only terms η_{s_1} with $s_1 < k$ appear. But then we have for $k \geq 1$

$$\begin{aligned}
(\eta\psi^p_{h'\upharpoonleft})_k &= -\eta_k + \sum_{\substack{r_0+s_1+r_1+s_2+r_2=k\\ r_0,r_1,r_2,r'_0,r'_1,r'_2\geq 0\\ s_1,s_2\geq 1}} \big((\hat{g})_{r_0,r'_0}\otimes \eta_{s_1}\otimes (\hat{f})_{r_1,r'_1}\otimes (h')_{s_2,1}\otimes (\hat{g})_{r_2,r'_2}\big)m_{r'_0+1+r'_1+1+r'_2,1} \\
&= \theta_k.
\end{aligned}$$

Hence we have constructed a graded linear map $\eta\colon TA \to B$ of degree p with $\eta\psi^p_{h'\upharpoonleft} = \theta$. Therefore $\psi^p_{h'\upharpoonleft}$ is surjective.

For $\psi^p_{h'\downharpoonleft}$, let $\eta\colon TA \to B$ be a graded linear map of degree p and let $h\colon TA \to TB$ be the unique (g,g)-coderivation of degree p such that $h\beta^p_{g,g} = \eta$. For $k \geq 1$ we have using Lemma 51

$$\begin{aligned}
&(\eta\psi^p_{h'\downharpoonleft})_k \\
&\quad = \iota_k(\eta\psi^p_{h'\downharpoonleft}) \\
&\quad = \iota_k(h\beta^p_{g,g}\psi^p_{h'\downharpoonleft}) \\
&\quad = \iota_k(h\Psi^p_{h'\downharpoonleft}\beta^p_{f,g})
\end{aligned}$$

$$
\begin{aligned}
&= \iota_k\big((h\Phi^{f,g}_{g,g} + (-1)^p (h' \otimes h) M^{p-1,(f,g)}_{2,1})\beta^p_{f,g}\big)\\
&= \iota_k(h\beta^p_{g,g}) + (-1)^p \iota_k\big((h' \otimes h) M^{p-1,(f,g)}_{2,1}\beta^p_{f,g}\big)\\
&= \iota_k \eta + (-1)^p \iota_k\big((h' \otimes h) M^{p-1,(f,g)}_{2,1}\big)\pi_1\\
&= \eta_k + (-1)^p \sum_{\substack{r_0+s_1+r_1+s_2+r_2=k\\ r_0,r_1,r_2,r_0',r_1',r_2'\geq 0\\ s_1,s_2\geq 1}} ((\hat{f})_{r_0,r_0'} \otimes (h')_{s_1,1} \otimes (\hat{g})_{r_1,r_1'} \otimes \eta_{s_2} \otimes (\hat{g})_{r_2,r_2'}) m_{r_0'+1+r_1'+1+r_2',1}.
\end{aligned}
$$

Injectivity of $\psi^p_{h'\lfloor}$. Suppose that $\eta\psi^p_{h'\lfloor} = 0$, i.e. $(\eta\psi^p_{h'\lfloor})_k = 0$ for $k \geq 1$. We show that $\eta_k = 0$ for $k \geq 1$ by induction on k. For $k = 1$ note that by the above formula $(\eta\psi^p_{h'\lfloor})_1 = \eta_1$, i.e. $\eta_1 = 0$. Now let $k > 1$ and suppose that $\eta_\ell = 0$ for $\ell < k$. But then the above formula for $(\eta\psi^p_{h'\lfloor})_k$ implies that $(\eta\psi^p_{h'\lfloor})_k = \eta_k$, since in the sum only terms η_{s_2} with $s_2 < k$ appear. Thus $\eta_k = 0$. Hence $\ker(\psi^p_{h'\lfloor}) = \{0\}$ and we conclude that $\psi^p_{h'\lfloor}$ is injective.
Surjectivity of $\psi^p_{h'\lfloor}$. Suppose given a graded linear map $\theta\colon TA \to B$ of degree p. We construct the components $\eta_k\colon A^{\otimes k} \to B$ of a graded linear map $\eta\colon TA \to B$ of degree p by the following recursive formula for $k \geq 1$.

$$
\eta_k := \theta_k - (-1)^p \sum_{\substack{r_0+s_1+r_1+s_2+r_2=k\\ r_0,r_1,r_2,r_0',r_1',r_2'\geq 0\\ s_1,s_2\geq 1}} ((\hat{f})_{r_0,r_0'} \otimes (h')_{s_1,1} \otimes (\hat{g})_{r_1,r_1'} \otimes \eta_{s_2} \otimes (\hat{g})_{r_2,r_2'}) m_{r_0'+1+r_1'+1+r_2',1}
$$

Note that in the above sum only terms η_{s_2} with $s_2 < k$ appear. But then we have for $k \geq 1$

$$
\begin{aligned}
&(\eta\psi^p_{h'\lfloor})_k\\
&\quad= \eta_k + (-1)^p \sum_{\substack{r_0+s_1+r_1+s_2+r_2=k\\ r_0,r_1,r_2,r_0',r_1',r_2'\geq 0\\ s_1,s_2\geq 1}} ((\hat{f})_{r_0,r_0'} \otimes (h')_{s_1,1} \otimes (\hat{g})_{r_1,r_1'} \otimes \eta_{s_2} \otimes (\hat{g})_{r_2,r_2'}) m_{r_0'+1+r_1'+1+r_2',1}\\
&\quad= \theta_k.
\end{aligned}
$$

Hence we have constructed a graded linear map $\eta\colon TA \to B$ of degree p with $\eta\psi^p_{h'\lfloor} = \theta$. Therefore $\psi^p_{h'\lfloor}$ is surjective. □

Lemma 61 *Being coderivation homotopic is an equivalence relation on the set* $\mathsf{dgCoalg}(TA, TB)$ *of morphisms of differential graded coalgebras from* TA *to* TB.

Proof. We have to show reflexivity, transitivity and symmetry.
We make use of Remark 59 without further comment, i.e. we use that an (f,g)-coderivation $h\colon TA \to TB$ of degree -1 is a homotopy if and only if $hM^{-1,(f,g)}_{1,1} = f - g$.
Reflexivity: The graded linear zero map $h = 0$ of degree -1 is an (f,g)-coderivation and satisfies $hM^{-1,(f,f)}_{1,1} = 0 = f - f$, hence is a homotopy from f to f.
Transitivity: Suppose given $f_0, f_1, f_2 \in \mathsf{dgCoalg}(TA, TB)$. Suppose there is a homotopy $h_1\colon TA \to TB$ from f_0 to f_1 and a homotopy $h_2\colon TA \to TB$ from f_1 to f_2. Define the (f_0, f_2)-coderivation $h\colon TA \to TB$ of degree -1 by

$$
h := h_1\Phi^{f_0,f_2}_{f_0,f_1} + h_2\Phi^{f_0,f_2}_{f_1,f_2} - (h_1 \otimes h_2)M^{-2,(f_0,f_2)}_{2,1}.
$$

Since M is a differential on $T\operatorname{Coder}(TA,TB)$ by Theorem 49, the tuple $(M_{k,1})_{k\geq 1}$ satisfies the Stasheff equations by Lemma 24.(1). In particular, we have

$$M_{1,1}M_{1,1}=0 \quad \text{and} \quad 0=M_{2,1}M_{1,1}+(\mathrm{id}\otimes M_{1,1}+M_{1,1}\otimes \mathrm{id})M_{2,1}. \qquad (*)$$

To show that h is a homotopy from f_0 to f_2, we have to show that $hM_{1,1}^{-1,(f_0,f_2)}=f_0-f_2$. We calculate using $(*)$, Lemma 54 and Lemma 56.

$$\begin{aligned}
hM_{1,1}^{-1,(f_0,f_2)} &= h_1\Phi^{f_0,f_2}_{f_0,f_1}M_{1,1}^{-1,(f_0,f_2)}+h_2\Phi^{f_0,f_2}_{f_1,f_2}M_{1,1}^{-1,(f_0,f_2)}-(h_1\otimes h_2)M_{2,1}^{-2,(f_0,f_2)}M_{1,1}^{-1,(f_0,f_2)}\\
&\overset{\mathrm{L}\,56}{=} h_1M_{1,1}^{-1,(f_0,f_1)}\Phi^{f_0,f_2}_{f_0,f_1}-\big(h_1\otimes(f_1-f_2)\big)M_{2,1}^{-1,(f_0,f_2)}\\
&\quad+h_2M_{1,1}^{-1,(f_1,f_2)}\Phi^{f_0,f_2}_{f_1,f_2}+\big((f_0-f_1)\otimes h_2\big)M_{2,1}^{-1,(f_0,f_2)}\\
&\quad-(h_1\otimes h_2)M_{2,1}^{-2,(f_0,f_2)}M_{1,1}^{-1,(f_0,f_2)}\\
&\overset{(*)}{=} h_1M_{1,1}^{-1,(f_0,f_1)}\Phi^{f_0,f_2}_{f_0,f_1}-\big(h_1\otimes(f_1-f_2)\big)M_{2,1}^{-1,(f_0,f_2)}\\
&\quad+h_2M_{1,1}^{-1,(f_1,f_2)}\Phi^{f_0,f_2}_{f_1,f_2}+\big((f_0-f_1)\otimes h_2\big)M_{2,1}^{-1,(f_0,f_2)}\\
&\quad+\big(h_1\otimes h_2M_{1,1}^{-1,(f_1,f_2)}\big)M_{2,1}^{-1,(f_0,f_2)}-\big(h_1M_{1,1}^{-1,(f_0,f_1)}\otimes h_2\big)M_{2,1}^{-1,(f_0,f_2)}\\
&= (f_0-f_1)\Phi^{f_0,f_2}_{f_0,f_1}-\big(h_1\otimes(f_1-f_2)\big)M_{2,1}^{-1,(f_0,f_2)}\\
&\quad+(f_1-f_2)\Phi^{f_0,f_2}_{f_1,f_2}+\big((f_0-f_1)\otimes h_2\big)M_{2,1}^{-1,(f_0,f_2)}\\
&\quad+\big(h_1\otimes(f_1-f_2)\big)M_{2,1}^{-1,(f_0,f_2)}-\big((f_0-f_1)\otimes h_2\big)M_{2,1}^{-1,(f_0,f_2)}\\
&= (f_0-f_1)\Phi^{f_0,f_2}_{f_0,f_1}+(f_1-f_2)\Phi^{f_0,f_2}_{f_1,f_2}\\
&\overset{\mathrm{L}\,54}{=} f_0-f_2
\end{aligned}$$

Hence h is a homotopy from f_0 to f_2.

Symmetry: Suppose given morphisms of differential graded coalgebras $f,g\in \mathsf{dgCoalg}(TA,TB)$ and a homotopy $h'\colon TA\to TB$ from f to g. In this case, we have the following isomorphism of differential $\mathbf{Z}$-graded modules from Lemma 60.

$$\begin{array}{llcl}
\Psi_{h'\upharpoonright}\colon & \operatorname{Coder}(TA,TB)^{(g,f)} & \longrightarrow & \operatorname{Coder}(TA,TB)^{(g,g)}\\
\Psi^p_{h'\upharpoonright}\colon & h & \longmapsto & -h(\Phi^{g,g}_{g,f})^p+(h\otimes h')M_{2,1}^{p-1,(g,g)}
\end{array}$$

Using Lemma 54 and Lemma 56 we have

$$\begin{aligned}
(g-f)\Psi_{h'\upharpoonright} &= -(g-f)\Phi^{g,g}_{g,f}+\big((g-f)\otimes h'\big)M_{2,1}^{p-1,(g,g)}\\
&\overset{\mathrm{L}\,56}{=} -(g-f)\Phi^{g,g}_{g,f}+h'\Phi^{g,g}_{f,g}M_{1,1}^{-1,(g,g)}-h'M_{1,1}^{-1,(f,g)}\Phi^{g,g}_{f,g}\\
&= -(g-f)\Phi^{g,g}_{g,f}-(f-g)\Phi^{g,g}_{f,g}+h'\Phi^{g,g}_{f,g}M_{1,1}^{-1,(g,g)}\\
&\overset{\mathrm{L}\,54}{=} -(g-g)+h'\Phi^{g,g}_{f,g}M_{1,1}^{-1,(g,g)}\\
&= h'\Phi^{g,g}_{f,g}M_{1,1}^{-1,(g,g)}.
\end{aligned}$$

Since $\Psi_{h'\upharpoonright}$ is an isomorphism, there is a unique (g,f)-coderivation $h\colon TA\to TB$ of degree -1 such that $h\Psi_{h'\upharpoonright}=h'\Phi^{g,g}_{f,g}$. But then we obtain with the calculation from above

$$hM_{1,1}^{-1,(g,f)}\Psi_{h'\upharpoonright}=h\Psi_{h'\upharpoonright}M_{1,1}^{-1,(g,g)}=h'\Phi^{g,g}_{f,g}M_{1,1}^{-1,(g,g)}=(g-f)\Psi_{h'\upharpoonright}.$$

Hence $hM_{1,1}^{-1,(g,f)}=g-f$, i.e. h is a homotopy from g to f. □

2.2.3 The homotopy categories of differential graded tensor coalgebras and of A_∞-algebras

Recall that by Definition 28 the category $\mathrm{A}_\infty\text{-}\mathsf{alg}$ of A_∞-algebras is equivalent to the full subcategory $\mathsf{dtCoalg}$ of $\mathsf{dgCoalg}$ consisting of the differential graded tensor coalgebras, cf. Definition 29. The equivalence is established by the full and faithful Bar-functor from Definition 28.

$$\mathrm{Bar}\colon \quad \mathrm{A}_\infty\text{-}\mathsf{alg} \longrightarrow \mathsf{dgCoalg}$$

Using this equivalence, we define A_∞-homotopy using the notion of coderivation homotopy from Definition 57.

Definition 62 Let $A = (A, (\mathsf{m}_k)_{k\geq 1})$ and $B = (B, (\mathsf{m}_k)_{k\geq 1})$ be A_∞-algebras.
Two morphisms of A_∞-algebras $f\colon A \to B$ and $g\colon A \to B$ are *homotopic* if the morphisms of differential graded coalgebras $\mathrm{Bar}\, f\colon TA^{[1]} \to TB^{[1]}$ and $\mathrm{Bar}\, g\colon TA^{[1]} \to TB^{[1]}$ are coderivation homotopic, cf. Definition 57.

Theorem 63

(1) *Being coderivation homotopic is a congruence on the category* $\mathsf{dtCoalg}$ *of differential graded tensor coalgebras.*

We obtain the homotopy category $\underline{\mathsf{dtCoalg}}$ *whose objects are differential graded tensor coalgebras and whose morphisms are equivalence classes of differential graded coalgebra morphisms under coderivation homotopy.*

For a morphism $f\colon TA \to TB$ *in* $\mathsf{dtCoalg}$ *we write* $[f]$ *for its equivalence class under this congruence. We call* $[f]$ *the* coderivation homotopy class *of* f.

(2) *Being homotopic is a congruence on the category* $\mathrm{A}_\infty\text{-}\mathsf{alg}$ *of* A_∞*-algebras.*

We obtain the homotopy category $\mathrm{A}_\infty\text{-}\underline{\mathsf{alg}}$ *whose objects are* A_∞*-algebras and whose morphisms are equivalence classes of morphisms of* A_∞*-algebras under homotopy.*

For a morphism $f\colon A \to B$ *in* $\mathrm{A}_\infty\text{-}\mathsf{alg}$ *we write* $[f]$ *for its homotopy class.*

(3) *The* Bar*-functor induces an equivalence*

$$\begin{aligned} \underline{\mathrm{Bar}}\colon \quad \mathrm{A}_\infty\text{-}\underline{\mathsf{alg}} &\longrightarrow \underline{\mathsf{dtCoalg}} \\ [f] &\longmapsto \underline{\mathrm{Bar}}[f] := [\mathrm{Bar}\, f]. \end{aligned}$$

In particular, the following diagram commutes where the vertical functors are the residue class functors that send a morphism to its homotopy class or coderivation homotopy class respectively.

$$\begin{array}{ccc} \mathrm{A}_\infty\text{-}\mathsf{alg} & \xrightarrow[\sim]{\mathrm{Bar}} & \mathsf{dtCoalg} \\ \downarrow & & \downarrow \\ \mathrm{A}_\infty\text{-}\underline{\mathsf{alg}} & \xrightarrow[\sim]{\underline{\mathrm{Bar}}} & \underline{\mathsf{dtCoalg}} \end{array}$$

Proof. (1) By Lemma 61 being coderivation homotopic is an equivalence relation and with Lemma 58 we conclude that it is a congruence.

(2) Suppose given two A_∞-algebras $A = (A, (\mathsf{m}_k)_{k\geq 1})$ and $B = (B, (\mathsf{m}_k)_{k\geq 1})$. By (1), coderivation homotopy is an equivalence relation on the set $\mathsf{dgCoalg}(\operatorname{Bar} A, \operatorname{Bar} B)$ of morphisms of differential graded coalgebras from $\operatorname{Bar} A$ to $\operatorname{Bar} B$. Since Bar is full and faithful, this implies that homotopy of morphisms of A_∞-algebras is an equivalence relation on the set $\mathrm{A}_\infty\text{-}\mathsf{alg}(A, B)$ of A_∞-algebra morphisms from A to B.

It remains to verify that homotopy is preserved under post- and precomposition. For this, let $A' = (A', (\mathsf{m}_k)_{k\geq 1})$, $A = (A, (\mathsf{m}_k)_{k\geq 1})$, $B = (B, (\mathsf{m}_k)_{k\geq 1})$ and $B' = (B', (\mathsf{m}_k)_{k\geq 1})$ be A_∞-algebras and let $s\colon A' \to A$, $f\colon A \to B$, $g\colon A \to B$ and $t\colon B \to B'$ be morphisms of A_∞-algebras such that f and g are homotopic. We have to show that sft and sgt are homotopic, i.e. we have to show that $\operatorname{Bar}(sft)$ and $\operatorname{Bar}(sgt)$ are coderivation homotopic. Since Bar is a functor we have $\operatorname{Bar}(sft) = (\operatorname{Bar} s)(\operatorname{Bar} f)(\operatorname{Bar} t)$ and $\operatorname{Bar}(sgt) = (\operatorname{Bar} s)(\operatorname{Bar} g)(\operatorname{Bar} t)$. By assumption $\operatorname{Bar} f$ and $\operatorname{Bar} g$ are coderivation homotopic, hence the assertion follows from (1).

(3) Let $f, g\colon A \to B$ be a morphisms of A_∞-algebras. By definition of the homotopy relation on $\mathrm{A}_\infty\text{-}\mathsf{alg}$, the morphisms f and g are homotopic if and only if $\operatorname{Bar} f$ and $\operatorname{Bar} g$ are coderivation homotopic. Moreover, as Bar is an equivalence between $\mathrm{A}_\infty\text{-}\mathsf{alg}$ and $\mathsf{dtCoalg}$, it is full and faithful. It follows that $\underline{\operatorname{Bar}}$ defines a full and faithful functor. Note that Bar and $\underline{\operatorname{Bar}}$ are the identity on objects. Thus $\underline{\operatorname{Bar}}$ is an equivalence. $\square$

Chapter 3

Homotopy equivalences

Let R be a commutative ring.

All modules are left R-modules, all linear maps between modules are R-linear maps, all tensor products of modules are tensor products over R.

Fix a grading category $\mathcal{Z}$. Unless stated otherwise, by *graded* we mean $\mathcal{Z}$*-graded*.

Our aim in this chapter is a characterisation of A_∞-homotopy equivalences, cf. Theorem 79. In the case where the ground ring R is a field, we recover Prouté's theorem which states that A_∞-quasiisomorphisms coincide with A_∞-homotopy equivalences, cf. Remark 80.

3.1 Homotopy equivalences of differential graded modules

3.1.1 The homotopy category of differential graded modules

Recall the abelian category dgMod of differential graded modules, cf. Definition 9.

Definition 64 Let $M = (M, d_M)$ and $N = (N, d_N)$ be differential graded modules.

(1) Let $f\colon M \to N$ and $g\colon M \to N$ be morphisms of differential graded modules.

A morphism f is called *null-homotopic* if there is a graded linear map $h\colon M \to N$ of degree -1 such that $f = hd_N + d_M h$. We call h a *homotopy*. We call the morphisms f and g *homotopic* if $f - g$ is null-homotopic.

Note that the set of null-homotopic maps is stable under sums, post- and precomposition, i.e. it forms an ideal $\mathcal{N} \subseteq \mathsf{dgMod}$.

(2) We denote by $\underline{\mathsf{dgMod}} = \mathsf{dgMod}/\mathcal{N}$ the *homotopy category of differential graded modules*. It has the same objects as dgMod, but morphisms are residue classes of morphisms of differential graded modules modulo null-homotopic maps, i.e.

$$\underline{\mathsf{dgMod}}(M, N) = \mathsf{dgMod}(M, N)/\{f \in \mathsf{dgMod}(M, N) : f \text{ is null-homotopic}\}.$$

We denote by $[f]$ the set of morphisms of differential graded modules that are homotopic to f, i.e. the residue class of f in $\underline{\mathsf{dgMod}}(M, N)$.

There is an additive residue class functor $\mathsf{dgMod} \to \underline{\mathsf{dgMod}}$, that is the identity on objects and sends a morphism f to its residue class $[f]$.

(3) A morphism of differential graded modules $f\colon M \to N$ is called a *homotopy equivalence*, if $[f]$ is an isomorphism in $\underline{\mathsf{dgMod}}$.

Note that f is a homotopy equivalence if and only if there is a morphism of differential graded modules $g\colon N \to M$ such that fg is homotopic to id_M and gf is homotopic to id_N.

(4) A differential graded module $M = (M, d_M)$ is called *split acyclic*, if the identity on M is homotopic to zero, i.e. if there is a graded linear map $h\colon M \to M$ of degree -1 such that $\mathrm{id}_M = hd_M + d_M h$. In this case, we say that h is a *contracting homotopy* on M.

3.1.2 Cones and factorisation of homotopy equivalences

Let (M, d_M) and (N, d_N) be differential graded modules.

Definition 65 Suppose given a morphism of differential graded modules $f\colon M \to N$. Consider the graded module $\mathrm{Cone}(f) := M^{[1]} \oplus N$ with the graded linear map $d_{\mathrm{Cone}(f)}$ of degree 1 given by

$$d_{\mathrm{Cone}(f)} := \begin{pmatrix} -d_M^{[1]} & f^{[1]} \\ 0 & d_N \end{pmatrix} : M^{[1]} \oplus N \to M^{[1]} \oplus N.$$

This is indeed a differential on $\mathrm{Cone}(f)$, since we have using that $fd_N = d_M f$ for $z \in \mathrm{Mor}(\mathcal{Z})$

$$d^z_{\mathrm{Cone}(f)} d^{z[1]}_{\mathrm{Cone}(f)} = \begin{pmatrix} -d_M^{z[1]} & f^{z[1]} \\ 0 & d_N^z \end{pmatrix} \begin{pmatrix} -d_M^{z[2]} & f^{z[2]} \\ 0 & d_N^{z[1]} \end{pmatrix} = \begin{pmatrix} d_M^{z[1]} d_M^{z[2]} & -d_M^{z[1]} f^{z[2]} + f^{z[1]} d_N^{z[1]} \\ 0 & d_N^z d_N^{z[1]} \end{pmatrix} = 0.$$

We obtain the differential graded module $\mathrm{Cone}(f) = (\mathrm{Cone}(f), d_{\mathrm{Cone}(f)})$, the *cone over f*. We also write $\mathrm{Cone}(M) := \mathrm{Cone}(\mathrm{id}_M)$.

Lemma 66 *The cone* $\mathrm{Cone}(M)$ *is split acyclic. Moreover, we have a morphism of differential graded modules* $i\colon M \to \mathrm{Cone}(M)$ *given by*

$$i := \begin{pmatrix} 0 & \mathrm{id}_M \end{pmatrix} : M \to M^{[1]} \oplus M.$$

Proof. To show that $\mathrm{Cone}(M)$ is split acyclic, let $h\colon \mathrm{Cone}(M) \to \mathrm{Cone}(M)$ be the graded linear map of degree -1 given by

$$h := \begin{pmatrix} 0 & 0 \\ \mathrm{id}_M & 0 \end{pmatrix} : M^{[1]} \oplus M \to M^{[1]} \oplus M.$$

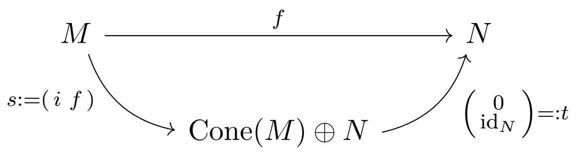
We claim that h defines a contracting homotopy on $\mathrm{Cone}(M)$. Indeed, we have for $z \in \mathrm{Mor}(\mathcal{Z})$

$$\begin{aligned} h^z d^{z[-1]}_{\mathrm{Cone}(M)} + d^z_{\mathrm{Cone}(M)} h^{z[1]} &= \begin{pmatrix} 0 & 0 \\ \mathrm{id}_M^z & 0 \end{pmatrix} \begin{pmatrix} -d_M^z & \mathrm{id}_M^z \\ 0 & d_M^{z[-1]} \end{pmatrix} + \begin{pmatrix} -d_M^{z[1]} & \mathrm{id}_M^{z[1]} \\ 0 & d_M^z \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \mathrm{id}_M^{z[1]} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ -d_M^z & \mathrm{id}_M^z \end{pmatrix} + \begin{pmatrix} \mathrm{id}_M^{z[1]} & 0 \\ d_M^z & 0 \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{id}_M^{z[1]} & 0 \\ 0 & \mathrm{id}_M^z \end{pmatrix} \\ &= \mathrm{id}^z_{\mathrm{Cone}(M)} . \end{aligned}$$

Thus $\mathrm{id}_M = hd_{\mathrm{Cone}(M)} + d_{\mathrm{Cone}(M)}h$, so $\mathrm{Cone}(M)$ is split acyclic.
Finally, to see that i is a morphism of differential graded modules, we have to verify that $d_M i = id_{\mathrm{Cone}(M)}$. But for $z \in \mathrm{Mor}(\mathcal{Z})$ we have

$$i^z d^z_{\mathrm{Cone}(M)} = \begin{pmatrix} 0 & \mathrm{id}^z_M \end{pmatrix} \begin{pmatrix} -d^{z[1]}_M & \mathrm{id}^{z[1]}_M \\ 0 & d^z_M \end{pmatrix} = \begin{pmatrix} 0 & d^z_M \end{pmatrix} = d^z_M \begin{pmatrix} 0 & \mathrm{id}^z_M \end{pmatrix} = d^z_M i^z. \qquad \square$$

Lemma 67 *Let $f\colon M \to N$ be a homotopy equivalence of differential graded modules. Let $i\colon M \to \mathrm{Cone}(M)$ be the morphism of differential graded modules from Lemma 66.*
Factorise f as in the following commutative diagram in dgMod*.*



Then both s and t are homotopy equivalences, s is a coretraction and t a retraction.

Proof. As t is the projection to a direct summand in dgMod, it is a retraction. Since $\mathrm{Cone}(M)$ is split acyclic, it is isomorphic to zero in the homotopy category. By additivity of the residue class functor $\mathsf{dgMod} \to \underline{\mathsf{dgMod}}$ it follows that $[t]$ is an isomorphism, i.e. f is a homotopy equivalence.
Since $[f] = [st] = [s][t]$ and $[t]$ is an isomorphism, it follows that $[s]$ is an isomorphism, i.e. s is a homotopy equivalence. It remains to show that s is a coretraction. Since f is a homotopy equivalence, there is a morphism of differential graded modules $g\colon M \to N$ and a homotopy $h\colon M \to M$ such that $fg - \mathrm{id}_M = hd_M + d_M h$. We define a graded linear map $r\colon M^{[1]} \oplus M \oplus N \to N$ of degree 0 by

$$r := \begin{pmatrix} -h^{[1]} \\ -hd_M - d_M h \\ g \end{pmatrix} : M^{[1]} \oplus M \oplus N \to M.$$

We claim that r is a morphism of differential graded modules from $\mathrm{Cone}(M) \oplus N \to M$. We have for $z \in \mathrm{Mor}(\mathcal{Z})$

$$\begin{aligned} d^z_{\mathrm{Cone}(M)\oplus N} r^{z[1]} &= \begin{pmatrix} -d^{z[1]}_M & \mathrm{id}^{z[1]}_M & 0 \\ 0 & d^z_M & 0 \\ 0 & 0 & d^z_N \end{pmatrix} \begin{pmatrix} -h^{z[2]} \\ -h^{z[1]} d^z_M - d^{z[1]}_M h^{z[2]} \\ g^{z[1]} \end{pmatrix} \\ &= \begin{pmatrix} -h^{z[1]} d^z_M \\ -d^z_M h^{z[1]} d^z_M \\ d^z_N g^{z[1]} \end{pmatrix} \\ &= \begin{pmatrix} -h^{z[1]} \\ -h^z d^{z[-1]}_M - d^z_M h^{z[1]} \\ g^z \end{pmatrix} d^z_M \\ &= r^z d^z_M. \end{aligned}$$

Hence r is a morphism of differential graded modules. Moreover, we have for $z \in \mathrm{Mor}(\mathcal{Z})$

$$s^z r^z = \begin{pmatrix} 0 & \mathrm{id}_M^z & f^z \end{pmatrix} \begin{pmatrix} -h^{z[1]} \\ -h^z d_M^{z[-1]} - d_M^z h^{z[1]} \\ g^z \end{pmatrix} = -h^z d_M^{z[-1]} - d_M^z h^{z[1]} + f^z g^z = \mathrm{id}_M^z .$$

Hence $sr = \mathrm{id}_M$, i.e. s is a coretraction in dgMod. □

3.2 A_∞-homotopy equivalences

Recall the full subcategory $\mathsf{dtCoalg}$ of $\mathsf{dgCoalg}$ of differential graded tensor coalgebras, cf. Definition 29. On $\mathsf{dtCoalg}$, we have the notion of coderivation homotopy, cf. Definition 57. Coderivation homotopy is a congruence and we have homotopy category $\underline{\mathsf{dtCoalg}}$, cf. Theorem 63.

A morphism $f\colon TA \to TB$ in $\mathsf{dtCoalg}$ is a homotopy equivalence in $\mathsf{dtCoalg}$ if its coderivation homotopy class $[f]\colon TA \to TB$ is an isomorphism in $\underline{\mathsf{dtCoalg}}$. Our goal is to characterise homotopy equivalences in $\mathsf{dtCoalg}$.

For this, certain morphisms in $\mathsf{dtCoalg}$ will be called acyclic cofibrations and acyclic fibrations, cf. Definition 69 below. However, we will not make use of the formal framework of a model category.

In [Lef03], a model structure is constructed on a certain full subcategory of $\mathsf{dgCoalg}$ when the ground ring R is a field. Restricted to $\mathsf{dtCoalg}$, the acyclic cofibrations and acyclic fibrations coincide with our definition below.

Some of the lemmas below are taken from [Lef03]. We reprove them here, to show that they still hold over a commutative ground ring.

3.2.1 Acyclic fibrations and cofibrations

Let $TA = (TA, \Delta, m)$, $TB = (TB, \Delta, m)$ and $TC = (TC, \Delta, m)$ be differential graded tensor coalgebras.

Lemma 68 *There is a functor*

$$\begin{aligned} V\colon \quad \mathsf{dtCoalg} &\longrightarrow \mathsf{dgMod} \\ (TA, \Delta, m) &\longmapsto (A, m_{1,1}) \\ (f\colon TA \to TB) &\longmapsto (f_{1,1}\colon A \to B). \end{aligned}$$

Note that $A = \ker(\Delta)$ by Lemma 19, i.e. we can recover A from TA.

The functor V induces a functor $\bar{V}\colon \underline{\mathsf{dtCoalg}} \to \underline{\mathsf{dgMod}}$ between the homotopy categories, given by $\bar{V}[f] = [Vf]$ for a differential graded coalgebra morphism $f\colon TA \to TB$.

In other words, the following diagram of functors commutes, where the vertical functors are the residue class functors.

$$\begin{array}{ccc} \mathsf{dtCoalg} & \xrightarrow{V} & \mathsf{dgMod} \\ \downarrow & & \downarrow \\ \underline{\mathsf{dtCoalg}} & \xrightarrow{\bar{V}} & \underline{\mathsf{dgMod}} \end{array}$$

Proof. Let (TA, Δ, m) be an object in dtCoalg. Then m is a coderivation, so by Lemma 23.(2) we have $Am \subseteq A$. Since $mm = 0$, we obtain $(mm)_{1,1} = m_{1,1}m_{1,1} = 0$. Hence $(A, m_{1,1})$ is a differential graded module.

We have $(\mathrm{id}_{TA})_{1,1} = \mathrm{id}_A$, hence $V(\mathrm{id}_{TA}) = \mathrm{id}_{V(TA)}$. Suppose given composable morphisms $f\colon TA \to TB$ and $g\colon TB \to TC$ in dtCoalg. We have $Af \subseteq B$ by Lemma 23.(1), hence we obtain $(fg)_{1,1} = f_{1,1}g_{1,1}$, i.e. $V(fg) = (Vf)(Vg)$. It follows that V is a functor.

To show the existence of $\bar{V}$, it suffices to show that V sends coderivation homotopic morphisms in dtCoalg to homotopic morphisms in dgMod. Suppose given morphisms $f\colon TA \to TB$ and $g\colon TA \to TB$ with a coderivation homotopy $h\colon TA \to TB$ between them, i.e. h is an (f, g)-coderivation of degree -1 that satisfies $f - g = mh + hm$, cf. Definition 57.

By Lemma 37 we have $h_{1,\ell} = 0$ for $\ell > 1$, so $Ah \subseteq B$. Hence $f - g = mh + hm$ implies that $f_{1,1} - g_{1,1} = m_{1,1}h_{1,1} + h_{1,1}m_{1,1}$. It follows that $h_{1,1}\colon A \to B$ is a homotopy of differential graded modules between $f_{1,1} = Vf$ and $g_{1,1} = Vg$. □

Definition 69 Let $f\colon TA \to TB$ be a morphism of differential graded coalgebras.

(1) The morphism f is called an *acyclic cofibration* if Vf is a coretraction and a homotopy equivalence of differential graded modules.

(2) The morphism f is called an *acyclic fibration* if Vf is a retraction and a homotopy equivalence of differential graded modules.

(3) The morphism f is called *strict* if $f_{k,1} = 0$ for $k \geq 2$.

Remark 70 Let $f\colon TA \to TB$ and $g\colon TB \to TC$ be morphisms of differential graded coalgebras.

(1) The morphism f is an isomorphism if and only if it is both an acyclic cofibration and an acyclic fibration.

(2) If f and g are acyclic cofibrations, then so is fg.

(3) If f and g are acyclic fibrations, then so is fg.

Proof. (1) If f is an isomorphism of differential graded coalgebras, then Vf is an isomorphism of differential graded modules, hence a retraction, a coretraction and a homotopy equivalence. It follows that f is both an acyclic cofibration and an acyclic fibration.

Conversely, let f be a morphism of differential graded coalgebras that is both an acyclic cofibration and an acyclic fibration. Then $Vf = f_{1,1}$ is a retraction and a coretraction of differential graded modules, hence an isomorphism. Now Lemma 26 implies that f is an isomorphism of graded coalgebras. Using Remark 17 we conclude that f is also an isomorphism of differential graded coalgebras.

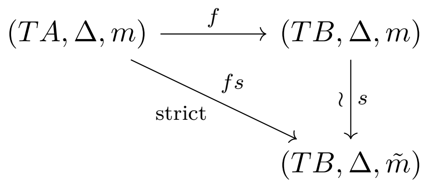
(2) Since the composite of two coretractions is again a coretraction, $V(fg) = (Vf)(Vg)$ is a coretraction of differential graded modules. Moreover, composites of homotopy equivalences are again homotopy equivalences. Hence $V(fg)$ is a coretraction and a homotopy equivalence, i.e. $V(fg)$ is an acyclic cofibration.

(3) Since the composite of two retractions is again a retraction, the same argument as in (2) shows that $V(fg)$ is an acyclic fibration. □

Lemma 71 (cf. [Lef03, Lemme 1.3.3.3])

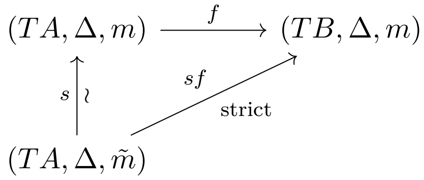
(1) *Let* $f\colon TA \to TB$ *be a morphism of differential graded coalgebras. Suppose that* $Vf = f_{1,1}$ *is a coretraction of graded modules, i.e. in* grMod.

Then there is a differential $\tilde{m}\colon TB \to TB$ *such that* $(TB, \Delta, \tilde{m})$ *is a differential graded tensor coalgebra and an isomorphism of differential graded coalgebras* $s\colon (TB, \Delta, m) \to (TB, \Delta, \tilde{m})$ *such that the composite* $fs\colon TA \to TB$ *is strict.*



(2) *Let* $f\colon TA \to TB$ *be a morphism of differential graded coalgebras. Suppose that* $Vf = f_{1,1}$ *is a retraction of graded modules, i.e. in* grMod.

Then there is a differential $\tilde{m}\colon TA \to TA$ *such that* $(TA, \Delta, \tilde{m})$ *is a differential graded tensor coalgebra and an isomorphism of differential graded coalgebras* $s\colon (TA, \Delta, \tilde{m}) \to (TA, \Delta, m)$ *such that the composite* $sf\colon TA \to TB$ *is strict.*



Proof. (1) By assumption, we may choose a graded linear map $g\colon B \to A$ of degree 0 such that $f_{1,1} g = \mathrm{id}_A$.

We construct the components $s_{k,1}\colon B^{\otimes k} \to B$ of a graded coalgebra morphism $s\colon TB \to TB$ for $k \geq 1$ recursively. For $k = 1$ we set $s_{1,1} = \mathrm{id}_B$. For $k \geq 2$ we set

$$s_{k,1} := -\sum_{i=1}^{k-1} g^{\otimes k} f_{k,i} s_{i,1}.$$

By Lemma 22.(1) this defines a graded coalgebra morphism $s\colon TB \to TB$. Using Lemma 26 we conclude that s is an isomorphism of graded coalgebras, since $s_{1,1}$ is an isomorphism of graded modules.

We define the differential $\tilde{m}$ on TB by $\tilde{m} := s^{-1} m s$. Then $\tilde{m}$ is an $(\mathrm{id}, \mathrm{id})$-coderivation of degree 1 by Lemma 36, i.e. it satisfies $\tilde{m}\Delta = \Delta(\mathrm{id} \otimes \tilde{m} + \tilde{m} \otimes \mathrm{id})$. Moreover, we have $\tilde{m}\tilde{m} = s^{-1} m s s^{-1} m s = s^{-1} m m s = 0$. Hence $(TB, \Delta, \tilde{m})$ is a differential graded coalgebra. Also note that $s\tilde{m} = s s^{-1} m s = ms$, thus s is an isomorphism of differential graded coalgebras.

It remains to show that the composite fs is strict, i.e. we have to show that $(fs)_{k,1} = 0$ for $k \geq 2$. Note that by Lemma 22.(1) we have $f_{k,k} = f_{1,1}^{\otimes k}$. We obtain

$$\begin{aligned}(fs)_{k,1} &\overset{\text{L 23}}{=} \sum_{i=1}^{k} f_{k,i} s_{i,1} = f_{k,k} s_{k,1} + \sum_{i=1}^{k-1} f_{k,i} s_{i,1} \\ &= -\sum_{i=1}^{k-1} f_{1,1}^{\otimes k} g^{\otimes k} f_{k,i} s_{i,1} + \sum_{i=1}^{k-1} f_{k,i} s_{i,1} = -\sum_{i=1}^{k-1} f_{k,i} s_{i,1} + \sum_{i=1}^{k-1} f_{k,i} s_{i,1} = 0.\end{aligned}$$

(2) By assumption, we may choose a graded linear map $g\colon B \to A$ of degree 0 such that $gf_{1,1} = \mathrm{id}_B$.
We construct the components $s_{k,1}\colon A^{\otimes k} \to A$ of a graded coalgebra morphism $s\colon TA \to TA$ for $k \geq 1$ recursively. For $k = 1$ we set $s_{1,1} = \mathrm{id}_A$. For $k \geq 2$ we set

$$s_{k,1} := -\sum_{i=2}^{k} \sum_{\substack{j_1+\ldots j_i=k\\ j_1,\ldots,j_i\geq 1}} (s_{j_1,1} \otimes \ldots \otimes s_{j_i,1}) f_{i,1} g.$$

By Lemma 22.(1) this defines a graded coalgebra morphism $s\colon TA \to TA$. In particular, we have for $k \geq 2$

$$s_{k,1} = -\sum_{i=2}^{k} s_{k,i} f_{i,1} g.$$

Using Lemma 26 we conclude that s is an isomorphism of graded coalgebras, as $s_{1,1}$ is an isomorphism of graded modules.
We define the differential $\tilde{m}$ on TA by $\tilde{m} := s^{-1} m s$. Then $\tilde{m}$ is an $(\mathrm{id}, \mathrm{id})$-coderivation of degree 1 by Lemma 36, i.e. it satisfies $\tilde{m}\Delta = \Delta(\mathrm{id} \otimes \tilde{m} + \tilde{m} \otimes \mathrm{id})$. Moreover, we have $\tilde{m}\tilde{m} = s^{-1} m s s^{-1} m s = s^{-1} m m s = 0$. Hence $(TA, \Delta, \tilde{m})$ is a differential graded coalgebra. Also note that $s\tilde{m} = s s^{-1} m s = ms$, thus s is an isomorphism of differential graded coalgebras.
It remains to show that the composite sf is strict, i.e. we have to show that $(sf)_{k,1} = 0$ for $k \geq 2$. We obtain

$$\begin{aligned}(sf)_{k,1} &\overset{\text{L 23}}{=} \sum_{i=1}^{k} s_{k,i} f_{i,1} = s_{k,1} f_{1,1} + \sum_{i=2}^{k} s_{k,i} f_{i,1}\\ &= -\sum_{i=2}^{k} s_{k,i} f_{i,1} g f_{1,1} + \sum_{i=2}^{k} s_{k,i} f_{i,1} = -\sum_{i=2}^{k} s_{k,i} f_{i,1} + \sum_{i=2}^{k} s_{k,i} f_{i,1} = 0.\end{aligned}$$

□

Lemma 72 *Let (M, d_M) and (N, d_N) be differential graded modules. Let $f\colon M \to N$ and $g\colon N \to M$ be morphisms of differential graded modules such that $fg = \mathrm{id}_M$ and gf is homotopic to id_N.*
Then there is a homotopy $h\colon N \to N$ from id_N to gf with $fh = 0$ and $hg = 0$.

Proof. By assumption, there is a homotopy $\tilde{h}\colon N \to N$ from id_N to gf, i.e. $\tilde{h}$ is a graded linear map of degree -1 with $\mathrm{id}_N - gf = d_N \tilde{h} + \tilde{h} d_N$. We set $h := (\mathrm{id}_N - gf)\tilde{h}(\mathrm{id}_N - gf)$. Then $h\colon N \to N$ is a graded linear map of degree -1. Since g and f are morphisms of differential graded modules with $fg = \mathrm{id}_M$, we have

$$\begin{aligned} d_N h + h d_N &= d_N(\mathrm{id}_N - gf)\tilde{h}(\mathrm{id}_N - gf) + (\mathrm{id}_N - gf)\tilde{h}(\mathrm{id}_N - gf) d_N\\ &= (\mathrm{id}_N - gf) d_N \tilde{h}(\mathrm{id}_N - gf) + (\mathrm{id}_N - gf)\tilde{h} d_N (\mathrm{id}_N - gf)\\ &= (\mathrm{id}_N - gf)(d_N \tilde{h} + \tilde{h} d_N)(\mathrm{id}_N - gf)\\ &= (\mathrm{id}_N - gf)(\mathrm{id}_N - gf)(\mathrm{id}_N - gf)\\ &= \mathrm{id}_N - 3gf + 3gfgf - gfgfgf\\ &= \mathrm{id}_N - gf.\end{aligned}$$

Hence h is a homotopy from id_N to gf that satisfies

$$fh = f(\mathrm{id}_N - gf)\tilde{h}(\mathrm{id}_N - gf) = (f - fgf)\tilde{h}(\mathrm{id}_N - gf) = (f - f)\tilde{h}(\mathrm{id}_N - gf) = 0$$

and

$$hg = (\mathrm{id}_N - gf)\tilde{h}(\mathrm{id}_N - gf)g = (\mathrm{id}_N - gf)\tilde{h}(g - gfg) = (\mathrm{id}_N - gf)\tilde{h}(g - g) = 0. \qquad \square$$

Lemma 73 *Let $g\colon TA \to TB$ be a morphism of graded coalgebras and let $k \geq 2$. Suppose that $(gm)_{\ell,1} = (mg)_{\ell,1}$ holds for $\ell < k$. Then the following equation of graded linear maps from $A^{\otimes k}$ to B of degree 2 holds.*

$$\sum_{j=1}^{k-1} m_{k,k} m_{k,j} g_{j,1} - \sum_{j=2}^{k} m_{k,k} g_{k,j} m_{j,1} = \sum_{j=2}^{k} g_{k,j} m_{j,1} m_{1,1} - \sum_{j=1}^{k-1} m_{k,j} g_{j,1} m_{1,1}$$

Proof. First note that $mm = 0$ implies that for $1 \leq j \leq k-1$ we have

$$0 = (mm)_{k,j} = \sum_{i=j}^{k} m_{k,i} m_{i,j}.$$

In particular, this gives

$$m_{k,k} m_{k,j} g_{j,1} = -\sum_{i=j}^{k-1} m_{k,i} m_{i,j} g_{j,1}.$$

By assumption, we know that $(gm)_{\ell,1} = (mg)_{\ell,1}$ for $1 \leq \ell \leq k-1$. Since gm and mg are (g,g)-coderivations by Lemma 36 we conclude using Lemma 38 that $(gm)_{r,s} = (mg)_{r,s}$ for $r, s \geq 1$ with $0 \leq r - s < k - 2$, i.e. we have

$$\sum_{i=s}^{r} g_{r,i} m_{i,s} = \sum_{i=s}^{r} m_{r,i} g_{i,s}.$$

In particular, we have for $2 \leq j \leq k$ that

$$m_{k,k} g_{k,j} m_{j,1} = -\sum_{i=j}^{k-1} m_{k,i} g_{i,j} m_{j,1} + \sum_{i=j}^{k} g_{k,i} m_{i,j} m_{j,1}.$$

Using these results we obtain

$$\begin{aligned}
&\sum_{j=1}^{k-1} m_{k,k} m_{k,j} g_{j,1} - \sum_{j=2}^{k} m_{k,k} g_{k,j} m_{j,1} \\
&= -\sum_{j=1}^{k-1} \sum_{i=j}^{k-1} m_{k,i} m_{i,j} g_{j,1} + \sum_{j=2}^{k} \sum_{i=j}^{k-1} m_{k,i} g_{i,j} m_{j,1} - \sum_{j=2}^{k} \sum_{i=j}^{k} g_{k,i} m_{i,j} m_{j,1} \\
&= -\sum_{i=1}^{k-1} \sum_{j=1}^{i} m_{k,i} m_{i,j} g_{j,1} + \sum_{i=2}^{k-1} \sum_{j=2}^{i} m_{k,i} g_{i,j} m_{j,1} - \sum_{i=2}^{k} \sum_{j=2}^{i} g_{k,i} m_{i,j} m_{j,1} \\
&= -m_{k,1} m_{1,1} g_{1,1} + \sum_{i=2}^{k-1} m_{k,i} \underbrace{\left(-\sum_{j=1}^{i} m_{i,j} g_{j,1} + \sum_{j=2}^{i} g_{i,j} m_{j,1} \right)}_{=-g_{i,1} m_{1,1}} - \sum_{i=2}^{k} g_{k,i} \underbrace{\left(\sum_{j=2}^{i} m_{i,j} m_{j,1} \right)}_{-m_{i,1} m_{1,1}}
\end{aligned}$$

$$= -m_{k,1}g_{1,1}m_{1,1} - \sum_{i=2}^{k-1} m_{k,i}g_{i,1}m_{1,1} + \sum_{i=2}^{k} g_{k,i}m_{i,1}m_{1,1}$$

$$= -\sum_{i=1}^{k-1} m_{k,i}g_{i,1}m_{1,1} + \sum_{i=2}^{k} g_{k,i}m_{i,1}m_{1,1}. \qquad \square$$

Lemma 74

(1) *Let $f\colon TA \to TB$ be a strict acyclic cofibration of differential graded tensor coalgebras. Then there is a differential graded coalgebra morphism $g\colon TB \to TA$ such that $fg = \mathrm{id}_{TA}$ and gf is coderivation homotopic to id_{TB}, where a coderivation homotopy $h\colon TB \to TB$ from id_{TB} to gf can be chosen such that $fh = 0$.*

(2) *Let $f\colon TA \to TB$ be a strict acyclic fibration of differential graded tensor coalgebras. Then there is a differential graded coalgebra morphism $g\colon TB \to TA$ such that $gf = \mathrm{id}_{TB}$ and fg is coderivation homotopic to id_{TA}, where a coderivation homotopy $h\colon TA \to TA$ from id_{TA} to fg can be chosen such that $hf = 0$.*

Proof. (1) Since f is an acyclic cofibration, there is a morphism of differential graded modules $\psi\colon B \to A$ such that $f_{1,1}\psi = \mathrm{id}_A$ and id_B is homotopic to $\psi f_{1,1}$. Recall that this means that $\psi m_{1,1} = m_{1,1}\psi$ holds and that there is a homotopy $\eta\colon B \to B$ such that $\mathrm{id}_B - \psi f_{1,1} = \eta m_{1,1} + m_{1,1}\eta$. Using Lemma 72 we can choose the homotopy η such that $f_{1,1}\eta = 0$.
To construct a graded coalgebra morphism $g\colon TA \to TB$, we give a recursive formula for its components $g_{k,1}\colon B^{\otimes k} \to A$. For $k = 1$ we set $g_{1,1} := \psi$. For $k \geq 2$ we set

$$g_{k,1} := \sum_{j=2}^{k} \sum_{\substack{u+v=k-1\\ u,v\geq 0}} \sum_{\substack{i_1+\ldots+i_j=k\\ i_1,\ldots,i_j\geq 1}} \big(\mathrm{id}^{\otimes u} \otimes \eta \otimes (g_{1,1}f_{1,1})^{\otimes v}\big)\big(g_{i_1,1} \otimes \ldots \otimes g_{i_j,1}\big)m_{j,1}$$
$$- \sum_{j=1}^{k-1} \sum_{\substack{u+v=k-1\\ u,v\geq 0}} (\mathrm{id}^{\otimes u} \otimes \eta \otimes (g_{1,1}f_{1,1})^{\otimes v})m_{k,j}g_{j,1}$$

By Lemma 22.(1) this defines a graded coalgebra morphism $g\colon TB \to TA$.
Similarly, to construct an (id, gf)-coderivation $h\colon TB \to TB$ of degree -1, we give a recursive formula for its components $h_{k,1}\colon B^{\otimes k} \to B$. For $k = 1$ we set $h_{1,1} := \eta$. For $k \geq 2$ we set

$$h_{k,1} := -\sum_{j=2}^{k} \sum_{\substack{u+v=k-1\\ u,v\geq 0}} \sum_{\substack{r+s+t=k\\ r+1+t'=j\\ r,t,t'\geq 0,\ s\geq 1}} \big(\mathrm{id}^{\otimes u} \otimes h_{1,1} \otimes (g_{1,1}f_{1,1})^{\otimes v}\big)\big(\mathrm{id}^{\otimes r} \otimes h_{s,1} \otimes (\widehat{gf})_{t,t'}\big)m_{j,1}$$
$$- \sum_{j=1}^{k-1} \sum_{\substack{u+v=k-1\\ u,v\geq 0}} (\mathrm{id}^{\otimes u} \otimes h_{1,1} \otimes (g_{1,1}f_{1,1})^{\otimes v})m_{k,j}h_{j,1}$$

By Lemma 37 this defines an (id, gf)-coderivation $h\colon TB \to TB$ of degree -1. Moreover, the same lemma implies that for $k, j \geq 1$

$$h_{k,j} = \sum_{\substack{r+s+t=k \\ r+1+t'=j \\ r,t,t' \geq 0,\ s \geq 1}} \mathrm{id}^{\otimes r} \otimes h_{s,1} \otimes (\widehat{gf})_{t,t'},$$

holds. In particular we have for $k = j$, using that $f_{k,k} = f_{1,1}^{\otimes k}$ from Lemma 22.(1)

$$h_{k,k} = \sum_{\substack{u+v=k-1 \\ u,v \geq 0}} \mathrm{id}^{\otimes u} \otimes h_{1,1} \otimes (g_{1,1} f_{1,1})^{\otimes v} = \sum_{\substack{u+v=k-1 \\ u,v \geq 0}} \mathrm{id}^{\otimes u} \otimes \eta \otimes (g_{1,1} f_{1,1})^{\otimes v}.$$

Moreover, Lemma 22.(1) implies that for $k, j \geq 1$

$$g_{k,j} = \sum_{\substack{i_1+\ldots+i_j=k \\ i_1,\ldots,i_j \geq 1}} g_{i_1,1} \otimes \ldots \otimes g_{i_j,1}.$$

Thus the defining formulas for $g_{k,1}$ and $h_{k,1}$ for $k \geq 2$ can be simplified to

$$g_{k,1} = \sum_{j=2}^{k} h_{k,k} g_{k,j} m_{j,1} - \sum_{j=1}^{k-1} h_{k,k} m_{k,j} g_{j,1}$$

and

$$h_{k,1} = -\sum_{j=2}^{k} h_{k,k} h_{k,j} m_{j,1} - \sum_{j=1}^{k-1} h_{k,k} m_{k,j} h_{j,1}.$$

We have to show that $fh = 0$, $fg = \mathrm{id}_{TA}$, $gm = mg$ and $\mathrm{id}_{TB} - gf = mh + hm$.

We show that $fh = 0$. Since fh is an (f, fgf)-coderivation by Lemma 36, it suffices to show that $(fh)_{k,1} = 0$ for $k \geq 1$ by Lemma 37. Since f is strict, we have $(fh)_{k,1} = f_{k,k} h_{k,1}$. But we have

$$\begin{aligned} f_{k,k} h_{k,k} &= \sum_{\substack{u+v=k-1 \\ u,v \geq 0}} f_{1,1}^{\otimes k} (\mathrm{id}^{\otimes u} \otimes h_{1,1} \otimes (g_{1,1} f_{1,1})^{\otimes v}) \\ &= \sum_{\substack{u+v=k-1 \\ u,v \geq 0}} f_{1,1}^{\otimes u} \otimes \underbrace{f_{1,1}\eta}_{=0} \otimes (f_{1,1} g_{1,1} f_{1,1})^{\otimes v} \\ &= 0. \end{aligned}$$

We conclude that

$$f_{k,k} h_{k,1} = -\sum_{j=2}^{k} f_{k,k} h_{k,k} h_{k,j} m_{j,1} - \sum_{j=1}^{k-1} f_{k,k} h_{k,k} m_{k,j} h_{j,1} = 0.$$

We show that $fg = \mathrm{id}_{TA}$. Since this is an equation of graded coalgebra morphisms, it suffices to show that $(fg)_{k,1} = (\mathrm{id}_{TA})_{k,1}$ for $k \geq 1$ by Lemma 22.(1). Hence we have to show that

$$(fg)_{k,1} = \begin{cases} \mathrm{id}_A & \text{if } k = 1 \\ 0 & \text{else} \end{cases}$$

for $k \geq 1$. For $k = 1$ we have $(fg)_{1,1} = f_{1,1}g_{1,1} = f_{1,1}\psi = \mathrm{id}_A$. For $k \geq 2$, note that $(fg)_{k,1} = f_{k,k}g_{k,1}$ since f is strict. We use that $fh = 0$, thus $(fh)_{k,k} = f_{k,k}h_{k,k} = 0$, and obtain

$$(fg)_{k,1} = f_{k,k}g_{k,1} = \sum_{j=2}^{k} f_{k,k}h_{k,k}g_{k,j}m_{j,1} - \sum_{j=1}^{k-1} f_{k,k}h_{k,k}m_{k,j}g_{j,1} = 0.$$

Claim: For $k \geq 1$ we have $\mathrm{id}_B^{\otimes k} - g_{k,k}f_{k,k} = h_{k,k}m_{k,k} + m_{k,k}h_{k,k}$. For $k = 1$ this follows by construction of $g_{1,1} = \psi$ and $h_{1,1} = \eta$. Now let $k \geq 2$. By Lemma 22.(2) we have

$$m_{k,k} = \sum_{\substack{r+t=k-1\\ r,t\geq 0}} \mathrm{id}^{\otimes r} \otimes m_{1,1} \otimes \mathrm{id}^{\otimes t} = \sum_{i=1}^{k} \mathrm{id}^{\otimes(i-1)} \otimes m_{1,1} \otimes \mathrm{id}^{\otimes(k-i)}$$

and we have seen above that

$$h_{k,k} = \sum_{\substack{u+v=k-1\\ u,v\geq 0}} \mathrm{id}^{\otimes u} \otimes h_{1,1} \otimes (g_{1,1}f_{1,1})^{\otimes v} = \sum_{j=1}^{k} \mathrm{id}^{\otimes(j-1)} \otimes h_{1,1} \otimes (g_{1,1}f_{1,1})^{\otimes(k-j)}$$

We calculate, starting from the right-hand side and paying attention to the Koszul sign rule.

$$\begin{aligned}
&h_{k,k}m_{k,k} + m_{k,k}h_{k,k}\\
&\quad= \sum_{j=1}^{k}\sum_{i=1}^{k} \big(\mathrm{id}^{\otimes(j-1)} \otimes h_{1,1} \otimes (g_{1,1}f_{1,1})^{\otimes(k-j)}\big)\big(\mathrm{id}^{\otimes(i-1)} \otimes m_{1,1} \otimes \mathrm{id}^{\otimes(k-i)}\big)\\
&\qquad+ \sum_{i=1}^{k}\sum_{j=1}^{k} \big(\mathrm{id}^{\otimes(i-1)} \otimes m_{1,1} \otimes \mathrm{id}^{\otimes(k-i)}\big)\big(\mathrm{id}^{\otimes(j-1)} \otimes h_{1,1} \otimes (g_{1,1}f_{1,1})^{\otimes(k-j)}\big)\\
&\quad= -\sum_{j=1}^{k}\sum_{i=1}^{j-1} \mathrm{id}^{\otimes(i-1)} \otimes m_{1,1} \otimes \mathrm{id}^{\otimes(j-i-1)} \otimes h_{1,1} \otimes (g_{1,1}f_{1,1})^{\otimes(k-j)}\\
&\qquad+ \sum_{j=1}^{k} \mathrm{id}^{\otimes(j-1)} \otimes h_{1,1}m_{1,1} \otimes (g_{1,1}f_{1,1})^{\otimes(k-j)}\\
&\qquad+ \sum_{j=1}^{k}\sum_{i=j+1}^{k} \mathrm{id}^{\otimes(j-1)} \otimes h_{1,1} \otimes (g_{1,1}f_{1,1})^{\otimes(i-j-1)} \otimes m_{1,1} \otimes (g_{1,1}f_{1,1})^{\otimes(k-i)}\\
&\qquad- \sum_{i=1}^{k}\sum_{j=1}^{i-1} \mathrm{id}^{\otimes(j-1)} \otimes h_{1,1} \otimes (g_{1,1}f_{1,1})^{\otimes(i-j-1)} \otimes m_{1,1} \otimes (g_{1,1}f_{1,1})^{\otimes(k-i)}\\
&\qquad+ \sum_{i=1}^{k} \mathrm{id}^{\otimes(i-1)} \otimes m_{1,1}h_{1,1} \otimes (g_{1,1}f_{1,1})^{\otimes(k-i)}\\
&\qquad+ \sum_{i=1}^{k}\sum_{j=i+1}^{k} \mathrm{id}^{\otimes(i-1)} \otimes m_{1,1} \otimes \mathrm{id}^{\otimes(j-i-1)} \otimes h_{1,1} \otimes (g_{1,1}f_{1,1})^{\otimes(k-j)}\\
&\quad= \sum_{i=1}^{k} \mathrm{id}^{\otimes(i-1)} \otimes (h_{1,1}m_{1,1} + m_{1,1}h_{1,1}) \otimes (g_{1,1}f_{1,1})^{\otimes(k-i)}\\
&\quad= \sum_{i=1}^{k} \mathrm{id}^{\otimes i} \otimes (g_{1,1}f_{1,1})^{\otimes(k-i)} - \sum_{i=1}^{k} \mathrm{id}^{\otimes(i-1)} \otimes (g_{1,1}f_{1,1})^{\otimes(k-i+1)}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{k} \mathrm{id}^{\otimes i} \otimes (g_{1,1} f_{1,1})^{\otimes (k-i)} - \sum_{i=0}^{k-1} \mathrm{id}^{\otimes i} \otimes (g_{1,1} f_{1,1})^{\otimes (k-i)} \\
&= \mathrm{id}^{\otimes k} - (g_{1,1} f_{1,1})^{\otimes k} \\
&= \mathrm{id}^{\otimes k} - g_{k,k} f_{k,k}.
\end{aligned}
$$

We show that $gm = mg$. This is an equation of (g,g)-coderivations by Lemma 36, so it suffices to show that $(gm)_{k,1} = (mg)_{k,1}$ for $k \geq 1$ by Lemma 37.
We use induction on k. For $k = 1$ we have $g_{1,1} = \psi$ and thus

$$
(gm)_{1,1} = g_{1,1} m_{1,1} = \psi m_{1,1} = m_{1,1} \psi = m_{1,1} g_{1,1} = (mg)_{1,1}.
$$

Now let $k \geq 2$ and suppose that $(gm)_{\ell,1} = (mg)_{\ell,1}$ holds for $1 \leq \ell \leq k-1$.
We have to show that $(gm)_{k,1} = (mg)_{k,1}$ for $k \geq 1$, i.e. we have to show that

$$
\sum_{j=1}^{k} g_{k,j} m_{j,1} = \sum_{j=1}^{k} m_{k,j} g_{j,1}
$$

or equivalently that for $k \geq 1$

$$
g_{k,1} m_{1,1} - m_{k,k} g_{k,1} = \sum_{j=1}^{k-1} m_{k,j} g_{j,1} - \sum_{j=2}^{k} g_{k,j} m_{j,1}. \tag{$*$}
$$

Since $fm = mf$ and using that f is strict, we have $f_{k,k} m_{k,j} = (fm)_{k,j} = (mf)_{k,j} = m_{k,j} f_{j,j}$. Since $fg = \mathrm{id}_{TA}$ and again using that f is strict, we have $f_{r,r} g_{r,s} = 0$ for $r \neq s$ and $f_{r,r} g_{r,s} = \mathrm{id}^{\otimes r}$ for $r = s$. We thus obtain

$$
\begin{aligned}
f_{k,k} \left(\sum_{j=1}^{k-1} m_{k,j} g_{j,1} - \sum_{j=2}^{k} g_{k,j} m_{j,1} \right) &= \sum_{j=1}^{k-1} f_{k,k} m_{k,j} g_{j,1} - \sum_{j=2}^{k} f_{k,k} g_{k,j} m_{j,1} \\
&= \sum_{j=1}^{k-1} m_{k,j} f_{j,j} g_{j,1} - \sum_{j=2}^{k} f_{k,k} g_{k,j} m_{j,1} \\
&= m_{k,1} - m_{k,1} \\
&= 0.
\end{aligned}
$$

Using this result, we start with the right-hand side in $(*)$ and the previous claim and obtain

$$
\begin{aligned}
\sum_{j=1}^{k-1} m_{k,j} g_{j,1} - \sum_{j=2}^{k} g_{k,j} m_{j,1} &= \left(\mathrm{id}^{\otimes k} - g_{k,k} f_{k,k} \right) \left(\sum_{j=1}^{k-1} m_{k,j} g_{j,1} - \sum_{j=2}^{k} g_{k,j} m_{j,1} \right) \\
&= \left(h_{k,k} m_{k,k} + m_{k,k} h_{k,k} \right) \left(\sum_{j=1}^{k-1} m_{k,j} g_{j,1} - \sum_{j=2}^{k} g_{k,j} m_{j,1} \right) \\
&= h_{k,k} \left(\sum_{j=1}^{k-1} m_{k,k} m_{k,j} g_{j,1} - \sum_{j=2}^{k} m_{k,k} g_{k,j} m_{j,1} \right) \\
&\quad + m_{k,k} \left(\sum_{j=1}^{k-1} h_{k,k} m_{k,j} g_{j,1} - \sum_{j=2}^{k} h_{k,k} g_{k,j} m_{j,1} \right)
\end{aligned}
$$

$$= \underbrace{h_{k,k}\left(\sum_{j=1}^{k-1} m_{k,k}m_{k,j}g_{j,1} - \sum_{j=2}^{k} m_{k,k}g_{k,j}m_{j,1}\right)}_{=:S} - m_{k,k}g_{k,1}$$

In order to show $(*)$ it remains to show that $S = g_{k,1}m_{1,1}$. But since

$$g_{k,1}m_{1,1} = h_{k,k}\left(\sum_{j=2}^{k} g_{k,j}m_{j,1}m_{1,1} - \sum_{j=1}^{k-1} m_{k,j}g_{j,1}m_{1,1}\right)$$

it suffices to show that

$$\sum_{j=1}^{k-1} m_{k,k}m_{k,j}g_{j,1} - \sum_{j=2}^{k} m_{k,k}g_{k,j}m_{j,1} = \sum_{j=2}^{k} g_{k,j}m_{j,1}m_{1,1} - \sum_{j=1}^{k-1} m_{k,j}g_{j,1}m_{1,1}.$$

But this equation holds by Lemma 73 using our induction hypothesis. Hence the verification of $gm = mg$ is completed.

We show that $\mathrm{id}_{TB} - gf = mh + hm$. Since $\mathrm{id}_{TB} - gf$ and $mh + hm = hM_{1,1}^{-1,(\mathrm{id},gf)}$ are (id, gf)-coderivations of degree 0 by Remark 59, it suffices to show that $(\mathrm{id}_{TB} - gf)_{k,1} = (mh + hm)_{k,1}$ for $k \geq 1$. We proceed using induction on k. The case $k = 1$ follows from the construction of $g_{1,1} = \psi$ and $h_{1,1} = \eta$. Now let $k \geq 2$. Since f is strict we have to show that

$$-g_{k,1}f_{1,1} = \sum_{j=1}^{k} m_{k,j}h_{j,1} + \sum_{j=1}^{k} h_{k,j}m_{j,1}. \qquad (*)$$

Since $fm = mf$ and using that f is strict we have $f_{k,k}m_{k,i} = (fm)_{k,i} = (mf)_{k,i} = m_{k,i}f_{i,i}$ for $k, i \geq 1$. Moreover, since $fh = 0$ we have $f_{j,j}h_{j,i} = 0$ for $j \geq i \geq 1$. Thus

$$\begin{aligned} f_{k,k}\left(\sum_{j=2}^{k} h_{k,j}m_{j,1} + \sum_{j=1}^{k-1} m_{k,j}h_{j,1}\right) &= \sum_{j=2}^{k} f_{k,k}h_{k,j}m_{j,1} + \sum_{j=1}^{k-1} f_{k,k}m_{k,j}h_{j,1} \\ &= \sum_{j=2}^{k} f_{k,k}h_{k,j}m_{j,1} + \sum_{j=1}^{k-1} m_{k,j}f_{j,j}h_{j,1} \\ &= 0. \end{aligned}$$

Hence the right-hand side of $(*)$ becomes with the previous claim

$$\begin{aligned} &\sum_{j=1}^{k} m_{k,j}h_{j,1} + \sum_{j=1}^{k} h_{k,j}m_{j,1} \\ &\quad = m_{k,k}h_{k,1} + h_{k,1}m_{1,1} + \sum_{j=2}^{k} h_{k,j}m_{j,1} + \sum_{j=1}^{k-1} m_{k,j}h_{j,1} \\ &\quad = m_{k,k}h_{k,1} + h_{k,1}m_{1,1} + \left(\mathrm{id}^{\otimes k} - g_{k,k}f_{k,k}\right)\left(\sum_{j=2}^{k} h_{k,j}m_{j,1} + \sum_{j=1}^{k-1} m_{k,j}h_{j,1}\right) \\ &\quad = m_{k,k}h_{k,1} + h_{k,1}m_{1,1} + \left(h_{k,k}m_{k,k} + m_{k,k}h_{k,k}\right)\left(\sum_{j=2}^{k} h_{k,j}m_{j,1} + \sum_{j=1}^{k-1} m_{k,j}h_{j,1}\right) \end{aligned}$$

$$
\begin{aligned}
&= m_{k,k}h_{k,1} + h_{k,1}m_{1,1} + h_{k,k}\left(\sum_{j=2}^{k} m_{k,k}h_{k,j}m_{j,1} + \sum_{j=1}^{k-1} m_{k,k}m_{k,j}h_{j,1}\right)\\
&\quad + m_{k,k}\underbrace{\left(\sum_{j=2}^{k} h_{k,k}h_{k,j}m_{j,1} + \sum_{j=1}^{k-1} h_{k,k}m_{k,j}h_{j,1}\right)}_{=-h_{k,1}}\\
&= h_{k,1}m_{1,1} + h_{k,k}\left(\sum_{j=2}^{k} m_{k,k}h_{k,j}m_{j,1} + \sum_{j=1}^{k-1} m_{k,k}m_{k,j}h_{j,1}\right)\\
&= -\sum_{j=2}^{k} h_{k,k}h_{k,j}m_{j,1}m_{1,1} - \sum_{j=1}^{k-1} h_{k,k}m_{k,j}h_{j,1}m_{1,1}\\
&\quad + \sum_{j=2}^{k} h_{k,k}m_{k,k}h_{k,j}m_{j,1} + \sum_{j=1}^{k-1} h_{k,k}m_{k,k}m_{k,j}h_{j,1} \qquad (**)
\end{aligned}
$$

We now continue with the left-hand side of $(*)$. Plugging in the defining formula for $g_{k,1}$ and using that $fm = mf$ we obtain

$$
-g_{k,1}f_{1,1} = -\sum_{j=2}^{k} h_{k,k}g_{k,j}f_{j,j}m_{j,1} + \sum_{j=1}^{k-1} h_{k,k}m_{k,j}g_{j,1}f_{1,1}
$$

Moreover, since by our induction hypothesis we have $(\mathrm{id}_{TB} - gf)_{\ell,1} = (hm + mh)_{\ell,1}$ for $1 \leq \ell \leq k-1$, Corollary 38 implies that for $r, s \geq 1$ with $0 \leq r - s < k-1$ also $(\mathrm{id}_{TB} - gf)_{r,s} = (hm + mh)_{r,s}$ holds, i.e. we have using that f is strict

$$
-g_{r,s}f_{s,s} = \begin{cases} -\mathrm{id}_B^{\otimes r} + h_{r,r}m_{r,r} + m_{r,r}h_{r,r} & \text{if } r = s \\ \sum_{i=s}^{r} h_{r,i}m_{i,s} + \sum_{i=s}^{r} m_{r,i}h_{i,s} & \text{else.} \end{cases}
$$

Thus we obtain

$$
\begin{aligned}
&-g_{k,1}f_{1,1}\\
&\quad = -\sum_{j=2}^{k} h_{k,k}g_{k,j}f_{j,j}m_{j,1} + \sum_{j=1}^{k-1} h_{k,k}m_{k,j}g_{j,1}f_{1,1}\\
&\quad = \left(\sum_{j=2}^{k-1} h_{k,k}\left(\sum_{i=j}^{k} h_{k,i}m_{i,j} + \sum_{i=j}^{k} m_{k,i}h_{i,j}\right)m_{j,1}\right) + h_{k,k}\Big(-\mathrm{id}_B^{\otimes k} + h_{k,k}m_{k,k} + m_{k,k}h_{k,k}\Big)m_{k,1}\\
&\qquad - \left(\sum_{j=2}^{k-1} h_{k,k}m_{k,j}\left(\sum_{i=1}^{j} h_{j,i}m_{i,1} + \sum_{i=1}^{j} m_{j,i}h_{i,1}\right)\right) - h_{k,k}m_{k,1}\Big(-\mathrm{id}_B + h_{1,1}m_{1,1} + m_{1,1}h_{1,1}\Big)\\
&\quad = \sum_{j=2}^{k}\sum_{i=j}^{k} h_{k,k}h_{k,i}m_{i,j}m_{j,1} + \sum_{j=2}^{k}\sum_{i=j}^{k} h_{k,k}m_{k,i}h_{i,j}m_{j,1}\\
&\qquad - \sum_{j=1}^{k-1}\sum_{i=1}^{j} h_{k,k}m_{k,j}h_{j,i}m_{i,1} - \sum_{j=1}^{k-1}\sum_{i=1}^{j} h_{k,k}m_{k,j}m_{j,i}h_{i,1}
\end{aligned}
$$

Now we consider the first and last double sum. Changing the order of summation and using that $mm = 0$ we obtain

$$\begin{aligned}\sum_{j=2}^{k}\sum_{i=j}^{k} h_{k,k}h_{k,i}m_{i,j}m_{j,1} - \sum_{j=1}^{k-1}\sum_{i=1}^{j} h_{k,k}m_{k,j}m_{j,i}h_{i,1}\\ &= \sum_{i=2}^{k}\sum_{j=2}^{i} h_{k,k}h_{k,i}m_{i,j}m_{j,1} - \sum_{i=1}^{k-1}\sum_{j=i}^{k-1} h_{k,k}m_{k,j}m_{j,i}h_{i,1}\\ &= -\sum_{i=2}^{k} h_{k,k}h_{k,i}m_{i,1}m_{1,1} + \sum_{i=1}^{k-1} h_{k,k}m_{k,k}m_{k,i}h_{i,1}.\end{aligned}$$

Now we consider the second and third double sum.

$$\begin{aligned}&\sum_{j=2}^{k}\sum_{i=j}^{k} h_{k,k}m_{k,i}h_{i,j}m_{j,1} - \sum_{j=1}^{k-1}\sum_{i=1}^{j} h_{k,k}m_{k,j}h_{j,i}m_{i,1}\\ &= \sum_{j=2}^{k}\sum_{i=j}^{k} h_{k,k}m_{k,i}h_{i,j}m_{j,1} - \sum_{i=1}^{k-1}\sum_{j=i}^{k-1} h_{k,k}m_{k,j}h_{j,i}m_{i,1}\\ &= \sum_{j=2}^{k-1}\sum_{i=j}^{k-1} h_{k,k}m_{k,i}h_{i,j}m_{j,1} + \sum_{j=2}^{k} h_{k,k}m_{k,k}h_{k,j}m_{j,1} - \sum_{i=1}^{k-1}\sum_{j=i}^{k-1} h_{k,k}m_{k,j}h_{j,i}m_{i,1}\\ &= \sum_{j=2}^{k} h_{k,k}m_{k,k}h_{k,j}m_{j,1} - \sum_{j=1}^{k-1} h_{k,k}m_{k,j}h_{j,1}m_{1,1}\end{aligned}$$

So altogether we obtain for the left-hand side of $(*)$

$$\begin{aligned}-g_{k,1}f_{1,1} = &-\sum_{i=2}^{k} h_{k,k}h_{k,i}m_{i,1}m_{1,1} + \sum_{i=1}^{k-1} h_{k,k}m_{k,k}m_{k,i}h_{i,1}\\ &+\sum_{j=2}^{k} h_{k,k}m_{k,k}h_{k,j}m_{j,1} - \sum_{j=1}^{k-1} h_{k,k}m_{k,j}h_{j,1}m_{1,1}.\end{aligned}$$

Comparing this with the right-hand side $(**)$ shows that $(*)$ holds true. This completes the verification of $\mathrm{id}_{TB} - gf = mh + hm$.

(2) Since f is an acyclic fibration, there is a morphism of differential graded modules $\psi\colon B \to A$ such that $\psi f_{1,1} = \mathrm{id}_B$ and id_A is homotopic to $f_{1,1}\psi$. Recall that this means that $\psi m_{1,1} = m_{1,1}\psi$ and that there is a homotopy $\eta\colon A \to A$ such that $\mathrm{id}_A - f_{1,1}\psi = m_{1,1}\eta + \eta m_{1,1}$. Using Lemma 72 we can choose the homotopy η such that $\eta f_{1,1} = 0$.

To construct a graded coalgebra morphism $g\colon TB \to TA$ we give a recursive formula for its components $g_{k,1}\colon B^{\otimes k} \to A$. For $k = 1$ we set $g_{1,1} := \psi$. For $k \geq 2$ we set

$$g_{k,1} := \sum_{j=1}^{k-1} m_{k,j}g_{j,1}\eta \; - \sum_{j=2}^{k} \sum_{\substack{i_1+\ldots+i_j=k\\ i_1,\ldots,i_j\geq 1}} (g_{i_1,1}\otimes \ldots \otimes g_{i_j,1})m_{j,1}\eta$$

By Lemma 22.(1) this defines a graded coalgebra morphism $g\colon TB \to TA$.

Similarly, to construct an (id, fg)-coderivation $h\colon TA \to TA$ of degree -1, we give a recursive formula for its components $h_{k,1}\colon A^{\otimes k} \to A$. For $k = 1$ we set $h_{1,1} := \eta$. For $k \geq 2$ we set

$$h_{k,1} := -\sum_{j=2}^{k} \sum_{\substack{r+s+t=k \\ r+1+t'=j \\ r,t,t'\geq 0,\ s\geq 1}} (\mathrm{id}^{\otimes r} \otimes h_{s,1} \otimes (\widehat{fg})_{t,t'}) m_{j,1}\eta \; - \; \sum_{j=1}^{k-1} m_{k,j} h_{j,1}\eta$$

By Lemma 37 this defines an (id, fg)-coderivation $h\colon TA \to TA$ of degree -1. The same lemma implies that for $k, j \geq 1$

$$h_{k,j} = \sum_{\substack{r+s+t=k \\ r+1+t'=j \\ r,t,t'\geq 0,\ s\geq 1}} \mathrm{id}^{\otimes r} \otimes h_{s,1} \otimes (\widehat{fg})_{t,t'},$$

holds. Moreover, Lemma 22.(1) implies that for $k, j \geq 1$

$$g_{k,j} = \sum_{\substack{i_1+\ldots+i_j=k \\ i_1,\ldots,i_j\geq 1}} g_{i_1,1} \otimes \ldots \otimes g_{i_j,1}.$$

Thus the defining formulas for $g_{k,1}$ and $h_{k,1}$ for $k \geq 2$ can be simplified to

$$g_{k,1} = \sum_{j=1}^{k-1} m_{k,j} g_{j,1} h_{1,1} - \sum_{j=2}^{k} g_{k,j} m_{j,1} h_{1,1}$$

and

$$h_{k,1} = -\sum_{j=2}^{k} h_{k,j} m_{j,1} h_{1,1} - \sum_{j=1}^{k-1} m_{k,j} h_{j,1} h_{1,1}.$$

We have to show that $hf = 0$, $gf = \mathrm{id}_{TB}$, $gm = mg$ and $\mathrm{id}_{TA} - fg = mh + hm$.

We show that $hf = 0$. Since hf is an (f, fgf)-coderivation by Lemma 36, it suffices to show that $(hf)_{k,1} = 0$ for $k \geq 1$ by Lemma 37. Since f is strict, we have $(hf)_{k,1} = h_{k,1} f_{1,1}$. Now recall that $h_{1,1} f_{1,1} = \eta f_{1,1} = 0$, which implies that

$$\begin{aligned} h_{k,1} f_{1,1} &= \left(-\sum_{j=2}^{k} h_{k,j} m_{j,1} h_{1,1} - \sum_{j=1}^{k-1} m_{k,j} h_{j,1} h_{1,1} \right) f_{1,1} \\ &= -\sum_{j=2}^{k} h_{k,j} m_{j,1} h_{1,1} f_{1,1} - \sum_{j=1}^{k-1} m_{k,j} h_{j,1} h_{1,1} f_{1,1} \\ &= 0. \end{aligned}$$

We show that $gf = \mathrm{id}_{TB}$. Since this is an equation of graded coalgebra morphisms, it suffices to show that $(gf)_{k,1} = (\mathrm{id}_{TB})_{k,1}$ for $k \geq 1$, cf. Lemma 22.(1). Hence we have to show that

$$(gf)_{k,1} = \begin{cases} \mathrm{id}_B & \text{if } k = 1 \\ 0 & \text{else} \end{cases}$$

for $k \geq 1$. For $k = 1$ we have $(gf)_{1,1} = g_{1,1}f_{1,1} = \psi f_{1,1} = \mathrm{id}_B$. For $k \geq 2$, note we have since f is strict that $(gf)_{k,1} = g_{k,1}f_{1,1}$. We use that $h_{1,1}f_{1,1} = \eta f_{1,1} = 0$ and obtain

$$
\begin{aligned}
g_{k,1}f_{1,1} &= \left(\sum_{j=1}^{k-1} m_{k,j}g_{j,1}h_{1,1} - \sum_{j=2}^{k} g_{k,j}m_{j,1}h_{1,1}\right)f_{1,1} \\
&= \sum_{j=1}^{k-1} m_{k,j}g_{j,1}h_{1,1}f_{1,1} - \sum_{j=2}^{k} g_{k,j}m_{j,1}h_{1,1}f_{1,1} \\
&= 0.
\end{aligned}
$$

We show that $gm = mg$. Since this is an equation of (g, g)-coderivations by Lemma 36, it suffices to show that $(gm)_{k,1} = (mg)_{k,1}$ for $k \geq 1$ by Lemma 37.
We use induction on k. For $k = 1$ we have $g_{1,1} = \psi$ and thus

$$(gm)_{1,1} = g_{1,1}m_{1,1} = \psi m_{1,1} = m_{1,1}\psi = m_{1,1}g_{1,1} = (mg)_{1,1}.$$

Now let $k \geq 2$ and suppose that $(gm)_{\ell,1} = (mg)_{\ell,1}$ holds for $1 \leq \ell \leq k - 1$.
We have to show that $(gm)_{k,1} = (mg)_{k,1}$ for $k \geq 1$, i.e. we have to show that

$$\sum_{j=1}^{k} g_{k,j}m_{j,1} = \sum_{j=1}^{k} m_{k,j}g_{j,1}$$

or equivalently that for $k \geq 1$

$$g_{k,1}m_{1,1} - m_{k,k}g_{k,1} = \sum_{j=1}^{k-1} m_{k,j}g_{j,1} - \sum_{j=2}^{k} g_{k,j}m_{j,1}. \tag{$*$}$$

Since $fm = mf$ and since f is strict we have $m_{j,1}f_{1,1} = (mf)_{j,1} = (fm)_{j,1} = f_{j,j}m_{j,1}$. Since $gf = \mathrm{id}_{TB}$ and again using that f is strict we have $g_{r,s}f_{s,s} = 0$ for $r \neq s$ and $g_{r,s}f_{s,s} = \mathrm{id}^{\otimes r}$ for $r = s$. We thus obtain

$$
\begin{aligned}
\left(\sum_{j=1}^{k-1} m_{k,j}g_{j,1} - \sum_{j=2}^{k} g_{k,j}m_{j,1}\right)f_{1,1} &= \sum_{j=1}^{k-1} m_{k,j}g_{j,1}f_{1,1} - \sum_{j=2}^{k} g_{k,j}m_{j,1}f_{1,1} \\
&= \sum_{j=1}^{k-1} m_{k,j}g_{j,1}f_{1,1} - \sum_{j=2}^{k} g_{k,j}f_{j,j}m_{j,1} \\
&= m_{k,1} - m_{k,1} \\
&= 0.
\end{aligned}
$$

Recall that $g_{1,1} = \psi$, $h_{1,1} = \eta$ and $\mathrm{id}_A - f_{1,1}g_{1,1} = m_{1,1}h_{1,1} + h_{1,1}m_{1,1}$ hold. We start with the right-hand side in $(*)$ and obtain

$$\begin{aligned}
\sum_{j=1}^{k-1} m_{k,j}g_{j,1} - \sum_{j=2}^{k} g_{k,j}m_{j,1} &= \Big(\sum_{j=1}^{k-1} m_{k,j}g_{j,1} - \sum_{j=2}^{k} g_{k,j}m_{j,1}\Big)\big(\mathrm{id}_A - f_{1,1}g_{1,1}\big) \\
&= \Big(\sum_{j=1}^{k-1} m_{k,j}g_{j,1} - \sum_{j=2}^{k} g_{k,j}m_{j,1}\Big)\big(m_{1,1}h_{1,1} + h_{1,1}m_{1,1}\big) \\
&= \Big(\sum_{j=1}^{k-1} m_{k,j}g_{j,1}m_{1,1} - \sum_{j=2}^{k} g_{k,j}m_{j,1}m_{1,1}\Big)h_{1,1} \\
&\quad + \Big(\sum_{j=1}^{k-1} m_{k,j}g_{j,1}h_{1,1} - \sum_{j=2}^{k} g_{k,j}m_{j,1}h_{1,1}\Big)m_{1,1} \\
&= \underbrace{\Big(\sum_{j=1}^{k-1} m_{k,j}g_{j,1}m_{1,1} - \sum_{j=2}^{k} g_{k,j}m_{j,1}m_{1,1}\Big)h_{1,1}}_{=:S} + g_{k,1}m_{1,1}.
\end{aligned}$$

Hence to show $(*)$ it remains to show that $S = -m_{k,k}g_{k,1}$. But since

$$\begin{aligned}
-m_{k,k}g_{k,1} &= \sum_{j=2}^{k} m_{k,k}g_{k,j}m_{j,1}h_{1,1} - \sum_{j=1}^{k-1} m_{k,k}m_{k,j}g_{j,1}h_{1,1} \\
&= \Big(\sum_{j=2}^{k} m_{k,k}g_{k,j}m_{j,1} - \sum_{j=1}^{k-1} m_{k,k}m_{k,j}g_{j,1}\Big)h_{1,1}
\end{aligned}$$

it suffices to show that

$$\sum_{j=1}^{k-1} m_{k,j}g_{j,1}m_{1,1} - \sum_{j=2}^{k} g_{k,j}m_{j,1}m_{1,1} = \sum_{j=2}^{k} m_{k,k}g_{k,j}m_{j,1} - \sum_{j=1}^{k-1} m_{k,k}m_{k,j}g_{j,1}.$$

But this equation holds by Lemma 73 using our induction hypothesis. Hence the verification of $gm = mg$ is complete.

We show that $\mathrm{id}_{TA} - fg = mh + hm$. Note that $\mathrm{id}_{TA} - fg$ and $mh + hm = hM_{1,1}^{-1,(\mathrm{id},fg)}$ are (id, fg)-coderivations of degree 0, cf. Remark 59. So it suffices to show that for $k \geq 1$ we have $(\mathrm{id}_{TA} - fg)_{k,1} = (mh + hm)_{k,1}$. We proceed using induction on k. For $k = 1$ note that we have $\mathrm{id}_A - f_{1,1}g_{1,1} = m_{1,1}h_{1,1} + h_{1,1}m_{1,1}$ by construction. Now let $k \geq 2$. Since f is strict we have to show that

$$-f_{k,k}g_{k,1} = \sum_{j=1}^{k} m_{k,j}h_{j,1} + \sum_{j=1}^{k} h_{k,j}m_{j,1}. \tag{$*$}$$

Since $fm = mf$ and using that f is strict we have $m_{j,1}f_{1,1} = (mf)_{j,1} = (fm)_{j,1} = f_{j,j}m_{j,1}$ for $j \geq 1$. Moreover, since $hf = 0$ we have $h_{j,i}f_{i,i} = 0$ for $j \geq i \geq 1$. Thus

$$\begin{aligned}
\Big(\sum_{j=2}^{k} h_{k,j}m_{j,1} + \sum_{j=1}^{k-1} m_{k,j}h_{j,1}\Big)f_{1,1} &= \sum_{j=2}^{k} h_{k,j}m_{j,1}f_{1,1} + \sum_{j=1}^{k-1} m_{k,j}h_{j,1}f_{1,1} \\
&= \sum_{j=2}^{k} h_{k,j}f_{j,j}m_{j,1} + \sum_{j=1}^{k-1} m_{k,j}h_{j,1}f_{1,1} \\
&= 0.
\end{aligned}$$

Hence the right-hand side of $(*)$ becomes

$$
\begin{aligned}
&\sum_{j=1}^{k} m_{k,j}h_{j,1} + \sum_{j=1}^{k} h_{k,j}m_{j,1} \\
&\quad = m_{k,k}h_{k,1} + h_{k,1}m_{1,1} + \sum_{j=2}^{k} h_{k,j}m_{j,1} + \sum_{j=1}^{k-1} m_{k,j}h_{j,1} \\
&\quad = m_{k,k}h_{k,1} + h_{k,1}m_{1,1} + \left(\sum_{j=2}^{k} h_{k,j}m_{j,1} + \sum_{j=1}^{k-1} m_{k,j}h_{j,1}\right)\left(\mathrm{id}_A - f_{1,1}g_{1,1}\right) \\
&\quad = m_{k,k}h_{k,1} + h_{k,1}m_{1,1} + \left(\sum_{j=2}^{k} h_{k,j}m_{j,1} + \sum_{j=1}^{k-1} m_{k,j}h_{j,1}\right)\left(m_{1,1}h_{1,1} + h_{1,1}m_{1,1}\right) \\
&\quad = m_{k,k}h_{k,1} + h_{k,1}m_{1,1} + \left(\sum_{j=2}^{k} h_{k,j}m_{j,1}m_{1,1} + \sum_{j=1}^{k-1} m_{k,j}h_{j,1}m_{1,1}\right)h_{1,1} \\
&\qquad + \underbrace{\left(\sum_{j=2}^{k} h_{k,j}m_{j,1}h_{1,1} + \sum_{j=1}^{k-1} m_{k,j}h_{j,1}h_{1,1}\right)}_{=-h_{k,1}} m_{1,1} \\
&\quad = m_{k,k}h_{k,1} + \left(\sum_{j=2}^{k} h_{k,j}m_{j,1}m_{1,1} + \sum_{j=1}^{k-1} m_{k,j}h_{j,1}m_{1,1}\right)h_{1,1} \\
&\quad = -\sum_{j=2}^{k} m_{k,k}h_{k,j}m_{j,1}h_{1,1} - \sum_{j=1}^{k-1} m_{k,k}m_{k,j}h_{j,1}h_{1,1} \\
&\qquad + \sum_{j=2}^{k} h_{k,j}m_{j,1}m_{1,1}h_{1,1} + \sum_{j=1}^{k-1} m_{k,j}h_{j,1}m_{1,1}h_{1,1} \qquad (**)
\end{aligned}
$$

We now continue with the left-hand side of $(*)$. Plugging in the defining formula for $g_{k,1}$ and using that $fm = mf$ we arrive at

$$
-f_{k,k}g_{k,1} = -\sum_{j=1}^{k-1} m_{k,j}f_{j,j}g_{j,1}h_{1,1} + \sum_{j=2}^{k} f_{k,k}g_{k,j}m_{j,1}h_{1,1}.
$$

By our induction hypothesis, we have $(\mathrm{id}_{TA} - fg)_{\ell,1} = (hm + mh)_{\ell,1}$ for $1 \le \ell \le k-1$. So Corollary 38 implies that for $r, s \ge 1$ with $0 \le r - s < k-1$ also $(\mathrm{id}_{TA} - fg)_{r,s} = (hm+mh)_{r,s}$ holds, i.e. we have

$$
-f_{r,r}g_{r,s} = \begin{cases} -\mathrm{id}_A^{\otimes r} + h_{r,r}m_{r,r} + m_{r,r}h_{r,r} & \text{if } r = s \\ \displaystyle\sum_{i=s}^{r} h_{r,i}m_{i,s} + \sum_{i=s}^{r} m_{r,i}h_{i,s} & \text{else.} \end{cases}
$$

Thus we obtain

$$
\begin{aligned}
&-f_{k,k}g_{k,1} \\
&\quad = -\sum_{j=1}^{k-1} m_{k,j}f_{j,j}g_{j,1}h_{1,1} + \sum_{j=2}^{k} f_{k,k}g_{k,j}m_{j,1}h_{1,1}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j=2}^{k-1} m_{k,j}\left(\sum_{i=1}^{j} h_{j,i}m_{i,1} + \sum_{i=1}^{j} m_{j,i}h_{i,1}\right)h_{1,1} + m_{k,1}\Big(-\operatorname{id}_A + h_{1,1}m_{1,1} + m_{1,1}h_{1,1}\Big)h_{1,1} \\
&\quad - \sum_{j=2}^{k-1}\left(\sum_{i=j}^{k} h_{k,i}m_{i,j} + \sum_{i=j}^{k} m_{k,i}h_{i,j}\right)m_{j,1}h_{1,1} - \Big(-\operatorname{id}_A^{\otimes k} + h_{k,k}m_{k,k} + m_{k,k}h_{k,k}\Big)m_{k,1}h_{1,1} \\
&= \sum_{j=1}^{k-1}\sum_{i=1}^{j} m_{k,j}h_{j,i}m_{i,1}h_{1,1} + \sum_{j=1}^{k-1}\sum_{i=1}^{j} m_{k,j}m_{j,i}h_{i,1}h_{1,1} \\
&\quad - \sum_{j=2}^{k}\sum_{i=j}^{k} h_{k,i}m_{i,j}m_{j,1}h_{1,1} - \sum_{j=2}^{k}\sum_{i=j}^{k} m_{k,i}h_{i,j}m_{j,1}h_{1,1}
\end{aligned}
$$

We consider the second and third double sum first. Changing the order of summation and using that $mm = 0$ we obtain

$$
\begin{aligned}
&\sum_{j=1}^{k-1}\sum_{i=1}^{j} m_{k,j}m_{j,i}h_{i,1}h_{1,1} - \sum_{j=2}^{k}\sum_{i=j}^{k} h_{k,i}m_{i,j}m_{j,1}h_{1,1} \\
&\qquad = \sum_{i=1}^{k-1}\sum_{j=i}^{k-1} m_{k,j}m_{j,i}h_{i,1}h_{1,1} - \sum_{i=2}^{k}\sum_{j=2}^{i} h_{k,i}m_{i,j}m_{j,1}h_{1,1} \\
&\qquad = -\sum_{i=1}^{k-1} m_{k,k}m_{k,i}h_{i,1}h_{1,1} + \sum_{i=2}^{k} h_{k,i}m_{i,1}m_{1,1}h_{1,1}.
\end{aligned}
$$

Now we consider the first and last double sum.

$$
\begin{aligned}
&\sum_{j=1}^{k-1}\sum_{i=1}^{j} m_{k,j}h_{j,i}m_{i,1}h_{1,1} - \sum_{j=2}^{k}\sum_{i=j}^{k} m_{k,i}h_{i,j}m_{j,1}h_{1,1} \\
&\quad = \sum_{j=1}^{k-1}\sum_{i=1}^{j} m_{k,j}h_{j,i}m_{i,1}h_{1,1} - \sum_{i=2}^{k}\sum_{j=2}^{i} m_{k,i}h_{i,j}m_{j,1}h_{1,1} \\
&\quad = \sum_{j=2}^{k-1}\sum_{i=2}^{j} m_{k,j}h_{j,i}m_{i,1}h_{1,1} + \sum_{j=1}^{k-1} m_{k,j}h_{j,1}m_{1,1}h_{1,1} - \sum_{i=2}^{k}\sum_{j=2}^{i} m_{k,i}h_{i,j}m_{j,1}h_{1,1} \\
&\quad = \sum_{j=1}^{k-1} m_{k,j}h_{j,1}m_{1,1}h_{1,1} - \sum_{j=2}^{k} m_{k,k}h_{k,j}m_{j,1}h_{1,1}
\end{aligned}
$$

So altogether we obtain for the left-hand side of $(*)$

$$
\begin{aligned}
-f_{k,k}g_{1,1} &= -\sum_{i=1}^{k-1} m_{k,k}m_{k,i}h_{i,1}h_{1,1} + \sum_{i=2}^{k} h_{k,i}m_{i,1}m_{1,1}h_{1,1} \\
&\quad + \sum_{j=1}^{k-1} m_{k,j}h_{j,1}m_{1,1}h_{1,1} - \sum_{j=2}^{k} m_{k,k}h_{k,j}m_{j,1}h_{1,1}.
\end{aligned}
$$

Comparing this with the right-hand side $(**)$ shows that $(*)$ holds true. This completes the verification of $\operatorname{id}_{TA} - gf = mh + hm$. □

Lemma 75

(1) *Let* $f\colon TA \to TB$ *be an acyclic cofibration of differential graded tensor coalgebras. Then there is a differential graded coalgebra morphism* $g\colon TB \to TA$ *such that* $fg = \mathrm{id}_{TA}$ *and* gf *is coderivation homotopic to* id_{TB}*.*

(2) *Let* $f\colon TA \to TB$ *be an acyclic fibration of differential graded tensor coalgebras. Then there is a differential graded coalgebra morphism* $g\colon TB \to TA$ *such that* $gf = \mathrm{id}_{TB}$ *and* fg *is coderivation homotopic to* id_{TA}*.*

Proof. Recall that we write $[\varphi]$ for the equivalence class of a differential graded coalgebra morphism $\varphi\colon TA \to TB$ under coderivation homotopy, i.e. $[\varphi]$ is the image of φ under the residue class functor $\mathsf{dtCoalg} \to \underline{\mathsf{dtCoalg}}$, cf. Theorem 63.

(1) Since f is an acyclic cofibration, Vf is a coretraction of differential graded modules, so in particular a coretraction of graded modules. Thus there is a differential graded coalgebra $(TB, \Delta, \tilde{m})$ and an isomorphism of differential graded coalgebras $s\colon (TB, \Delta, m) \to (TB, \Delta, \tilde{m})$ such that fs is strict, cf. Lemma 71.(1). Now fs is also an acyclic cofibration, cf. Remark 70. By Lemma 74.(1) there is a differential graded coalgebra morphism $\tilde{g}\colon (TB, \Delta, \tilde{m}) \to (TA, \Delta, m)$ with $fs\tilde{g} = \mathrm{id}_{TA}$ and $\tilde{g}fs$ coderivation homotopic to id_{TB}, i.e. $[\tilde{g}fs] = [\mathrm{id}_{TB}]$. Let $g := s\tilde{g}$. Then $fg = fs\tilde{g} = \mathrm{id}_{TA}$ and

$$[gf] = [s\tilde{g}f] = [s\tilde{g}fss^{-1}] = [s][\tilde{g}fs][s^{-1}] = [s][\mathrm{id}_{TB}][s^{-1}] = [ss^{-1}] = [\mathrm{id}_{TB}].$$

Hence gf is coderivation homotopic to id_{TB}.

(2) Since f is an acyclic fibration, Vf is a retraction of differential graded modules, so in particular a retraction of graded modules. Thus there is a differential graded coalgebra $(TA, \Delta, \tilde{m})$ and an isomorphism of differential graded coalgebras $s\colon (TA, \Delta, \tilde{m}) \to (TA, \Delta, m)$ such that sf is strict, cf. Lemma 71.(2). Now sf is also an acyclic fibration, cf. Remark 70. By Lemma 74.(2) there is a differential graded coalgebra morphism $\tilde{g}\colon (TB, \Delta, m) \to (TA, \Delta, \tilde{m})$ with $\tilde{g}sf = \mathrm{id}_{TB}$ and $sf\tilde{g}$ coderivation homotopic to id_{TA}, i.e. $[sf\tilde{g}] = [\mathrm{id}_{TA}]$. Let $g := \tilde{g}s$. Then $gf = \tilde{g}sf = \mathrm{id}_{TB}$ and

$$[fg] = [f\tilde{g}s] = [s^{-1}sf\tilde{g}s] = [s^{-1}][sf\tilde{g}][s] = [s^{-1}][\mathrm{id}_{TA}][s] = [s^{-1}s] = [\mathrm{id}_{TA}].$$

Hence fg is coderivation homotopic to id_{TA}. □

3.2.2 Products

Let $TA = (TA, \Delta, m^A)$ and $TB = (TB, \Delta, m^B)$ be differential graded tensor coalgebras.

Lemma 76 *Let* $C := A \oplus B$ *be the direct sum as graded modules. Consider the tensor coalgebra* (TC, Δ) *over* C*. Let* $p_{TA}\colon TC \to TA$ *be the strict graded coalgebra morphism such that* $p_A = (p_{TA})_{1,1}\colon C \to A$ *is the projection to* A *and let* $p_{TB}\colon TC \to TB$ *be the strict graded coalgebra morphism such that* $p_B = (p_{TB})_{1,1}\colon C \to B$ *is the projection to* B*. Let* $i_A\colon A \to C$ *and* $i_B\colon B \to C$ *be the graded linear inclusion map. Let* $m^C\colon TC \to TC$ *be the coderivation of degree 1 with*

$$m^C_{k,1} := p_A^{\otimes k} m^A_{k,1} i_A + p_B^{\otimes k} m^B_{k,1} i_B,$$

for $k \geq 1$, cf. Lemma 22.(2).
Then (TC, Δ, m^C) is the product of TA and TB in $\mathsf{dtCoalg}$ *with projections p_{TA} and p_{TB}. In particular, the functor $V\colon$ $\mathsf{dtCoalg} \to \mathsf{dgMod}$ from Lemma 68 preserves finite products.*

Proof. We have to show that (TC, Δ, m^C) is a differential graded tensor coalgebra, i.e. an object in $\mathsf{dtCoalg}$. For this, we have to verify that m^C is a differential. By Lemma 24.(1), it suffices to verify that $(m^C_{k,1})_{k\geq 1}$ satisfies the Stasheff equations. But we have for $k \geq 1$

$$\begin{aligned}
&\sum_{\substack{r+s+t=k\\ r,t\geq 0,\, s\geq 1}} (\mathrm{id}^{\otimes r} \otimes m^C_{s,1} \otimes \mathrm{id}^{\otimes t}) m^C_{r+1+t,1} \\
&\quad = \sum_{\substack{r+s+t=k\\ r,t\geq 0,\, s\geq 1}} \Big(\mathrm{id}^{\otimes r} \otimes \Big(p_A^{\otimes s} m^A_{s,1} i_A + p_B^{\otimes s} m^B_{s,1} i_B\Big) \otimes \mathrm{id}^{\otimes t}\Big) \\
&\qquad\quad \cdot \Big(p_A^{\otimes(r+1+t)} m^A_{r+1+t,1} i_A + p_B^{\otimes(r+1+t)} m^B_{r+1+t,1} i_B\Big) \\
&\quad = \sum_{\substack{r+s+t=k\\ r,t\geq 0,\, s\geq 1}} \Big(p_A^{\otimes r} \otimes \Big(p_A^{\otimes s} m^A_{s,1} i_A + p_B^{\otimes s} m^B_{s,1} i_B\Big) p_A \otimes p_A^{\otimes t}\Big) m^A_{r+1+t,1} i_A \\
&\qquad + \sum_{\substack{r+s+t=k\\ r,t\geq 0,\, s\geq 1}} \Big(p_B^{\otimes r} \otimes \Big(p_A^{\otimes s} m^A_{s,1} i_A + p_B^{\otimes s} m^B_{s,1} i_B\Big) p_B \otimes p_B^{\otimes t}\Big) m^B_{r+1+t,1} i_B \\
&\quad = \sum_{\substack{r+s+t=k\\ r,t\geq 0,\, s\geq 1}} p_A^{\otimes k} \Big(\mathrm{id}^{\otimes r} \otimes m^A_{s,1} \otimes \mathrm{id}^{\otimes t}\Big) m^A_{r+1+t,1} i_A \\
&\qquad + \sum_{\substack{r+s+t=k\\ r,t\geq 0,\, s\geq 1}} p_B^{\otimes k} \Big(\mathrm{id}^{\otimes r} \otimes m^B_{s,1} \otimes \mathrm{id}^{\otimes t}\Big) m^B_{r+1+t,1} i_B \\
&\quad = 0.
\end{aligned}$$

Hence (TC, Δ, m^C) is a differential graded tensor coalgebra, thus an object in $\mathsf{dtCoalg}$. The projection morphisms p_{TA} and p_{TB} are morphisms of differential graded coalgebras, since for $k \geq 1$ we have

$$(m^C p_{TA})_{k,1} = m^C_{k,1} (p_{TA})_{1,1} = m^C_{k,1} p_A = p_A^{\otimes k} m^A_{k,1} = (p_{TA})_{k,k} m^A_{k,1} = (p_{TA} m^A)_{k,1}$$

and

$$(m^C p_{TB})_{k,1} = m^C_{k,1} (p_{TB})_{1,1} = m^C_{k,1} p_B = p_B^{\otimes k} m^B_{k,1} = (p_{TB})_{k,k} m^B_{k,1} = (p_{TB} m^B)_{k,1}.$$

We claim that TC with the two morphisms p_{TA} and p_{TB} is a product of TA and TB in $\mathsf{dtCoalg}$. For this, let (TD, Δ, m^D) be another object in $\mathsf{dtCoalg}$ and let $u\colon TD \to TA$ and $v\colon TD \to TB$ be morphisms of differential graded coalgebras. We have to show that there is a unique morphism of differential graded coalgebras $w\colon TD \to TC$ with $wp_{TA} = u$ and $wp_{TB} = v$.

Uniqueness. A morphism of differential graded coalgebras $w\colon TD \to TC$ is uniquely determined by its components $w_{k,1}\colon D \to C$ for $k \geq 1$, cf. Lemma 22.(1). But since p_{TA} and p_{TB} are strict and their $(1,1)$-components are the projections p_A onto A and p_B onto B, we conclude from

$wp_{TA} = u$ that $w_{k,1}p_A = (wp_{TA})_{k,1} = u_{k,1}$ and from $wp_{TB} = v$ that $w_{k,1}p_B = (wp_{TB})_{k,1} = v_{k,1}$. Since $C = A \oplus B$, it follows that the components $w_{k,1}$ are uniquely determined.

Existence. Define a graded coalgebra morphism $w\colon TD \to TC$ by its components

$$w_{k,1} := u_{k,1}i_A + v_{k,1}i_B$$

for $k \geq 1$, cf. Lemma 22.(1). Since p_{TA} and p_{TB} are strict, we have for $k \geq 1$

$$(wp_{TA})_{k,1} = w_{k,1}(p_{TA})_{1,1} = (u_{k,1}i_A + v_{k,1}i_B)p_A = u_{k,1},$$

hence $wp_{TA} = u$. On the other hand, we have

$$(wp_{TB})_{k,1} = w_{k,1}(p_{TB})_{1,1} = (u_{k,1}i_A + v_{k,1}i_B)p_B = v_{k,1},$$

hence $wp_{TB} = v$. It remains to show that w is a morphism of differential graded coalgebras. For this, we have to show by Lemma 24.(2) that for $k \geq 1$ the following equation holds.

$$\sum_{\substack{r+s+t=k\\ r,t\geq 0,\, s\geq 1}} (\mathrm{id}^{\otimes r} \otimes m^D_{s,1} \otimes \mathrm{id}^{\otimes t}) w_{r+1+t,1} = \sum_{\ell=1}^{k} \sum_{\substack{j_1+\ldots+j_\ell=k\\ j_1,\ldots,j_\ell\geq 1}} (w_{j_1,1} \otimes \ldots \otimes w_{j_\ell,1}) m^C_{\ell,1}.$$

But starting with the right-hand side we obtain using that u and v are morphisms of differential graded coalgebras together with Lemma 24.(2)

$$\begin{aligned}
&\sum_{\ell=1}^{k} \sum_{\substack{j_1+\ldots+j_\ell=k\\ j_1,\ldots,j_\ell\geq 1}} (w_{j_1,1} \otimes \ldots \otimes w_{j_\ell,1}) m^C_{\ell,1}\\
&\quad= \sum_{\ell=1}^{k} \sum_{\substack{j_1+\ldots+j_\ell=k\\ j_1,\ldots,j_\ell\geq 1}} (w_{j_1,1} \otimes \ldots \otimes w_{j_\ell,1}) (p_A^{\otimes \ell} m^A_{\ell,1} i_A + p_B^{\otimes \ell} m^B_{\ell,1} i_B)\\
&\quad= \sum_{\ell=1}^{k} \sum_{\substack{j_1+\ldots+j_\ell=k\\ j_1,\ldots,j_\ell\geq 1}} ((w_{j_1,1}p_A) \otimes \ldots \otimes (w_{j_\ell,1}p_A)) m^A_{\ell,1} i_A\\
&\qquad+ \sum_{\ell=1}^{k} \sum_{\substack{j_1+\ldots+j_\ell=k\\ j_1,\ldots,j_\ell\geq 1}} ((w_{j_1,1}p_B) \otimes \ldots \otimes (w_{j_\ell,1}p_B)) m^B_{\ell,1} i_B\\
&\quad= \sum_{\ell=1}^{k} \sum_{\substack{j_1+\ldots+j_\ell=k\\ j_1,\ldots,j_\ell\geq 1}} (u_{j_1,1} \otimes \ldots \otimes u_{j_\ell,1}) m^A_{\ell,1} i_A\\
&\qquad+ \sum_{\ell=1}^{k} \sum_{\substack{j_1+\ldots+j_\ell=k\\ j_1,\ldots,j_\ell\geq 1}} (v_{j_1,1} \otimes \ldots \otimes v_{j_\ell,1}) m^B_{\ell,1} i_B\\
&\quad= \sum_{\substack{r+s+t=k\\ r,t\geq 0,\, s\geq 1}} (\mathrm{id}^{\otimes r} \otimes m^D_{s,1} \otimes \mathrm{id}^{\otimes t}) u_{r+1+t,1} i_A\\
&\qquad+ \sum_{\substack{r+s+t=k\\ r,t\geq 0,\, s\geq 1}} (\mathrm{id}^{\otimes r} \otimes m^D_{s,1} \otimes \mathrm{id}^{\otimes t}) v_{r+1+t,1} i_B
\end{aligned}$$

$$= \sum_{\substack{r+s+t=k \\ r,t\geq 0,\, s\geq 1}} (\mathrm{id}^{\otimes r} \otimes m^D_{s,1} \otimes \mathrm{id}^{\otimes t}) w_{r+1+t,1}$$

Thus w is a morphism of differential graded coalgebras with $wp_{TA} = u$ and $wp_{TB} = v$.
Finally, to see that V preserves products, recall that $V(TA) = (A, m^A_{1,1})$. For TC, we have $V(TC) = (C, m^C_{1,1})$ with $C = A \oplus B$ and graded modules and

$$m^C_{1,1} = p_A m^A_{1,1} i_A + p_B m^B_{1,1} i_B = \begin{pmatrix} m^A_{1,1} & 0 \\ 0 & m^B_{1,1} \end{pmatrix} : A \oplus B \to A \oplus B.$$

Moreover, for the projection morphisms we have $V(p_{TA}) = (p_{TA})_{1,1} = p_A$ and $V(p_{TB}) = p_B$. It follows that $V(TC)$ is a direct sum, i.e. a product, of $V(TA)$ and $V(TB)$ in dgMod. □

3.2.3 Factorisation

Let $TA = (TA, \Delta, m)$ and $TB = (TB, \Delta, m)$ be differential graded tensor coalgebras.

Lemma 77 (cf. [Lef03, Lemme 1.3.3.2]) *Suppose that the differential m on TB satisfies $m_{k,1} = 0$ for $k \geq 2$. Suppose that $(B, m_{1,1})$ is split acyclic. Let $\varphi\colon A \to B$ be a morphism of differential graded modules between $V(TA) = (A, m_{1,1})$ and $V(TB) = (B, m_{1,1})$.*
Then there exists a morphism of differential graded coalgebras $f\colon TA \to TB$ with $f_{1,1} = \varphi$.

Proof. Since $(B, m_{1,1})$ is split acyclic, there is a graded linear map $\eta\colon B \to B$ of degree -1 such that $\mathrm{id}_B = \eta m_{1,1} + m_{1,1}\eta$.
We define a graded coalgebra morphism $f\colon TA \to TB$ by its components $f_{k,1}$ for $k \geq 1$ recursively. For $k = 1$, set $f_{1,1} := \varphi$. For $k \geq 2$, set

$$f_{k,1} := m_{k,1}\varphi\eta.$$

This defines a graded coalgebra morphism by Lemma 22.(1). We have to show that f is a morphism of differential graded coalgebra, i.e. we have to verify that $fm = mf$. For this, it suffices to show that $(fm)_{k,1} = (mf)_{k,1}$ by Lemma 37. Since $m_{k,1} = 0$ for $k \geq 2$ on TB, we have to show that

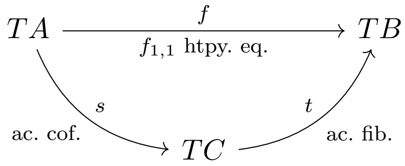
$$f_{k,1}m_{1,1} = \sum_{\ell=1}^{k} m_{k,\ell} f_{\ell,1}.$$

However, the right-hand side becomes, using $mm = 0$

$$\begin{aligned}
\sum_{\ell=1}^{k} m_{k,\ell} f_{\ell,1} &= m_{k,1}\varphi + \sum_{\ell=2}^{k} m_{k,\ell} m_{\ell,1}\varphi\eta \\
&= m_{k,1}\varphi - m_{k,1}m_{1,1}\varphi\eta \\
&= m_{k,1}\varphi - m_{k,1}\varphi m_{1,1}\eta \\
&= m_{k,1}\varphi(\mathrm{id}_B - m_{1,1}\eta) \\
&= m_{k,1}\varphi\eta m_{1,1} \\
&= f_{k,1}m_{1,1}.
\end{aligned}$$

Thus f is a morphism of differential graded coalgebras. □

Lemma 78 *Let $f\colon TA \to TB$ be a morphism of differential graded coalgebras such that $Vf = f_{1,1}\colon A \to B$ is a homotopy equivalence of differential graded modules.*
Then there is a differential graded tensor coalgebra $TC = (TC, \Delta, m)$, an acyclic cofibration $s\colon TA \to TC$ and an acyclic fibration $t\colon TC \to TB$ of differential graded tensor coalgebras such that $f = st$ holds.

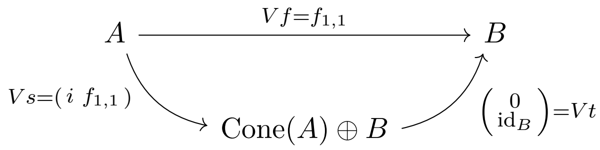


Proof. Let Cone(A) be the cone of the differential graded module $(A, m_{1,1})$. Then Cone(A) is a split acyclic differential graded module and we have the morphism of differential graded modules $i\colon A \to \mathrm{Cone}(A)$, cf. Lemma 66. Let $(T\,\mathrm{Cone}(A), \Delta, m)$ be the differential graded coalgebra in dtCoalg with $m_{k,1} = 0$ for $k \geq 2$ and $m_{1,1}$ being the differential on Cone(A), cf. Lemma 22.(2) and Lemma 24.(1).

By Lemma 77 there is a morphism of differential graded coalgebras $j\colon TA \to T\,\mathrm{Cone}(A)$ such that $j_{1,1} = i$.

Now let $TC = T\,\mathrm{Cone}(A) \times TB$ be a product of $T\,\mathrm{Cone}(A)$ and TB in dtCoalg, cf. Lemma 76. Denote by $p_1\colon TC \to T\,\mathrm{Cone}(A)$ and $p_2\colon TC \to TB$ the projection morphisms. By the universal property of the product, there is a morphism of differential graded coalgebras $s\colon TA \to TC$ with $sp_1 = j$ and $sp_2 = f$. Let $t = p_2$ be the projection morphism. Then we have $f = st$.

Since the functor $V\colon$ dtCoalg $\to$ dgMod from Lemma 68 preserves finite products (cf. Lemma 76), applying the functor to the equation $f = st$ yields the following commutative diagram.



Lemma 67 implies that Vs and Vt are homotopy equivalences of differential graded modules, Vs is a coretraction and Vt is a retraction. That is, s is an acyclic cofibration of differential graded tensor coalgebras and Vt is an acyclic fibration of differential graded tensor coalgebras. □

3.2.4 A characterisation of homotopy equivalences

Let (TA, Δ, m) and (TB, Δ, m) be differential graded tensor coalgebras.

Theorem 79 *Let $f\colon TA \to TB$ be a morphism of differential graded coalgebras.*
Then f is a homotopy equivalence of differential graded coalgebras if and only if $Vf = f_{1,1}$ is a homotopy equivalence of differential graded modules.
In other words, the functor $\bar{V}\colon \underline{\mathsf{dtCoalg}} \to \underline{\mathsf{dgMod}}$ from Lemma 68 reflects isomorphisms.

Proof. Recall that we denote by $[f]$ the homotopy class of f under coderivation homotopy.

If f is a homotopy equivalence of differential graded coalgebras, then $[f]$ is an isomorphism and hence $\bar{V}[f]$ is an isomorphism. By construction of the functors V and $\bar{V}$ we conclude that Vf is a homotopy equivalence of differential graded modules.

Conversely, suppose that $Vf = f_{1,1}$ is a homotopy equivalence of differential graded modules. By Lemma 78 we can factorise f into an acyclic cofibration $s\colon TA \to TC$ and an acyclic fibration $t\colon TC \to TB$ of differential graded tensor coalgebras, i.e. we have $f = st$.

By Lemma 75 both s and t are homotopy equivalences of differential graded tensor coalgebras, i.e. $[s]$ and $[t]$ are isomorphisms. But then also $[f] = [st] = [s][t]$ is an isomorphism, i.e. f is a homotopy equivalence of differential graded coalgebras. □

Remark 80 Suppose that R is a field. In this case, a morphism of differential graded modules is a homotopy equivalence if and only if its a quasiisomorphism.

Recall that an A_∞-quasiisomorphism is an A_∞-isomorphism $f = (f_k)_{k\geq 1}$ such that f_1 is a quasiisomorphism of complexes, cf. Definition 13 and Remark 15.

Hence Theorem 80 implies that over a ground *field* an A_∞-morphism is an A_∞-quasiisomorphism if and only if it is an A_∞-homotopy equivalence.

In this form, the theorem is due to Prouté [Pro84, Théorème 4.27], see also [Kel01, Theorem in section 3.7] and [Sei08, Corollary 1.14].

Remark 81 In general, the functor $\bar{V}\colon \underline{\mathsf{dtCoalg}} \to \underline{\mathsf{dgMod}}$ is neither full nor faithful.

Proof. Let $R = K$ be a field of characteristic $\operatorname{char} K \neq 2$. Let the grading category $\mathcal{Z} = \mathbf{Z}$ be given by the integers.

To show that in general $\bar{V}$ is not full, consider the graded module A with $A^z = K$ for $z = -1$ and $A^z = 0$ for $z \in \mathbf{Z} \setminus \{-1\}$.

Let $m\colon TA \to TA$ be the coderivation of degree 1 with $m_{k,1} = 0$ for $k \neq 2$ and

$$\begin{array}{rccl} m_{2,1}\colon & A\otimes A & \longrightarrow & A \\ m^z_{2,1}\colon & (a\otimes b) & \longmapsto & \begin{cases} ab \in A^{-1} & \text{if } z=-2 \text{ and } \lfloor a\rfloor = \lfloor b\rfloor = -1 \\ 0 & \text{else.}\end{cases}\end{array}$$

This defines a coderivation by Lemma 22.(2). We claim that m is a differential, i.e. we claim that $mm = 0$. By Lemma 24.(1) it suffices to verify that

$$0 = \sum_{\substack{r+s+t=k \\ r,t\geq 0,\, s\geq 1}} (\mathrm{id}_A^{\otimes r} \otimes m_{s,1} \otimes \mathrm{id}_A^{\otimes t}) m_{r+1+t,1}$$

holds for $k \geq 1$. However, since $m_{k,1} = 0$ for $k \neq 2$ it suffices to consider the case $k = 3$. In this case, we have to verify that

$$0 = (m_{2,1} \otimes \mathrm{id}_A) m_{2,1} + (\mathrm{id}_A \otimes m_{2,1}) m_{2,1}.$$

Let $z \in \mathbf{Z}$ and $a \otimes b \otimes c \in (A \otimes A \otimes A)^z$. Since $m^z_{2,1} = 0$ for $z \neq -2$, we only have to consider the case $a, b, c \in A^{-1}$. Then we have

$$\begin{aligned}(a \otimes b \otimes c)\big((m_{2,1} \otimes \mathrm{id}_A)m_{2,1} + (\mathrm{id}_A \otimes m_{2,1})m_{2,1}\big)\\ &= -((a \otimes b)m_{2,1} \otimes c)m_{2,1} + (a \otimes (b \otimes c)m_{2,1})m_{2,1}\\ &= -(ab \otimes c)m_{2,1} + (a \otimes bc)m_{2,1}\\ &= -abc + abc\\ &= 0.\end{aligned}$$

Hence m is a differential, i.e. $TA = (TA, \Delta, m)$ is an object in dtCoalg. Note that TA is the Bar construction of the unital differential graded algebra K concentrated in degree 0.

Let $f\colon TA \to TA$ be a morphism of differential graded coalgebras. Then f is uniquely determined by its components $f_{k,1}\colon A^{\otimes k} \to A$, which are graded linear maps of degree 0. For degree reasons, the components $f_{k,1}$ have to be zero for $k \geq 2$, as a non-zero element of $A^{\otimes k}$ has degree $-k$, but A only has non-zero elements in degree -1. Moreover, by Lemma 24.(2) they satisfy

$$\sum_{\substack{r+s+t=k\\ r,t\geq 0,\, \geq 1}} (\mathrm{id}_A^{\otimes r} \otimes m_{s,1} \otimes \mathrm{id}_A^{\otimes t})f_{r+1+t,1} = \sum_{r=1}^{k} \sum_{\substack{i_1+\ldots+i_r=k\\ i_1,\ldots,i_r\geq 1}} (f_{i_1,1} \otimes \ldots \otimes f_{i_r,1})m_{r,1}$$

for $k \geq 1$. In particular, they have to satisfy

$$m_{2,1}f_{1,1} = (f_{1,1} \otimes f_{1,1})m_{2,1}.$$

But then there is no morphism of differential graded coalgebras f such that $Vf = f_{1,1} = 2 \cdot \mathrm{id}_A$. Since $m_{1,1} = 0$, the (differential graded module) homotopy class of Vf is given by $[Vf] = \{Vf\}$, hence there is no morphism of differential graded coalgebras f such that $\bar{V}[f] = [Vf] = [2 \cdot \mathrm{id}_A]$. It follows that $\bar{V}$ is not full.

To show that in general $\bar{V}$ is not faithful, we construct a differential graded coalgebra TA, i.e. an object in dtCoalg, and a morphism of differential graded coalgebras $f\colon TA \to TA$ such that $\bar{V}[f] = \bar{V}[\mathrm{id}_{TA}]$, but $[f] \neq [\mathrm{id}_{TA}]$.

Consider the associative two-dimensional K-algebra $K[x]/(x^2)$. Let A be the $\mathbf{Z}$-graded module with $A^{-2} = A^{-1} = K[x]/(x^2)$ and $A^k = 0$ for $k \in \mathbf{Z} \setminus \{-1, -2\}$.

Let $m\colon TA \to TA$ be the coderivation of degree 1 with $m_{k,1} = 0$ for $k \neq 2$ and

$$\begin{aligned} m_{2,1}\colon \quad & A \otimes A \longrightarrow A\\ m_{2,1}^z\colon \quad & (a \otimes b) \longmapsto \begin{cases} xab \in A^{-1} & \text{if } z = -2 \text{ and } \lfloor a \rfloor = \lfloor b \rfloor = -1\\ 0 & \text{else.}\end{cases}\end{aligned}$$

This defines a coderivation by Lemma 22.(2). We claim that m is a differential, i.e. $mm = 0$. By Lemma 24.(1) it suffices to show that for $k \geq 1$

$$0 = \sum_{\substack{r+s+t=k\\ r,t\geq 0,\, s\geq 1}} (\mathrm{id}_A^{\otimes r} \otimes m_{s,1} \otimes \mathrm{id}_A^{\otimes t})m_{r+1+t,1}$$

holds. However, since $m_{k,1} = 0$ for $k \neq 2$, it suffices to consider the case $k = 3$. In this case, we have to verify that

$$0 = (m_{2,1} \otimes \mathrm{id}_A)m_{2,1} + (\mathrm{id}_A \otimes m_{2,1})m_{2,1}.$$

Let $z \in \mathbf{Z}$ and $a \otimes b \otimes c \in (A \otimes A \otimes A)^z$. Then

$$\begin{aligned}&(a \otimes b \otimes c)\big((m_{2,1} \otimes \mathrm{id}_A)m_{2,1} + (\mathrm{id}_A \otimes m_{2,1})m_{2,1}\big)\\ &\quad = (-1)^{\lfloor c \rfloor}((a \otimes b)m_{2,1} \otimes c)m_{2,1} + (a \otimes (b \otimes c)m_{2,1})m_{2,1}\end{aligned}$$

Since $m_{2,1}^z = 0$ for $z \neq -2$, we only have to consider the case $a, b, c \in A^{-1}$. We obtain

$$\begin{aligned}&(a \otimes b \otimes c)\big((m_{2,1} \otimes \mathrm{id}_A)m_{2,1} + (\mathrm{id}_A \otimes m_{2,1})m_{2,1}\big)\\ &\quad = -((a \otimes b)m_{2,1} \otimes c)m_{2,1} + (a \otimes (b \otimes c)m_{2,1})m_{2,1}\\ &\quad = -(axb \otimes c)m_{2,1} + (a \otimes xbc)m_{2,1}\\ &\quad = -abcx^2 + abcx^2\\ &\quad = 0.\end{aligned}$$

It follows that $mm = 0$. Note that TA is the Bar construction of a non-unital differential graded algebras concentrated in degrees 0 and -1.

Let f be the morphism of graded coalgebras with $f_{1,1} = \mathrm{id}_A$, $f_{k,1} = 0$ for $k \geq 3$ and

$$\begin{array}{rccl} f_{2,1}\colon & A \otimes A & \longrightarrow & A \\ f_{2,1}^z\colon & a \otimes b & \longmapsto & \begin{cases} ab \in A^{-2} & \text{if } z = -2 \text{ and } \lfloor a \rfloor = \lfloor b \rfloor = -1 \\ 0 & \text{else.} \end{cases} \end{array}$$

Recall that all components are graded linear maps of degree 0. This defines a morphism of graded coalgebras by Lemma 22.(1). We claim that f is a morphism of differential graded coalgebras. By Lemma 24.(2) it suffices to show that for $k \geq 1$

$$\sum_{\substack{r+s+t=k\\ r,t \geq 0,\, \geq 1}} (\mathrm{id}_A^{\otimes r} \otimes m_{s,1} \otimes \mathrm{id}_A^{\otimes t})f_{r+1+t,1} = \sum_{r=1}^{k} \sum_{\substack{i_1+\ldots+i_r=k\\ i_1,\ldots,i_r \geq 1}} (f_{i_1,1} \otimes \ldots \otimes f_{i_r,1})m_{r,1}$$

holds. For $k = 1$, both sides of the equation equals zero since $m_{1,1} = 0$. For $k = 2$, we have to show that

$$m_{2,1}f_{1,1} = (f_{1,1} \otimes f_{1,1})m_{2,1},$$

which is fulfilled since $f_{1,1} = \mathrm{id}_{TA}$. For $k = 3$, we have to show that

$$(m_{2,1} \otimes \mathrm{id}_A + \mathrm{id}_A \otimes m_{2,1})f_{2,1} = (f_{2,1} \otimes f_{1,1} + f_{1,1} \otimes f_{2,1})m_{2,1}.$$

The right-hand side is zero since $f_{2,1}^{-1} = 0$. For the left-hand side a similar calculation as for $mm = 0$ above shows that it also equals zero, i.e. for $a, b, c \in A^{-1}$ we have

$$\begin{aligned}&(a \otimes b \otimes c)\big((m_{2,1} \otimes \mathrm{id}_A)f_{2,1} + (\mathrm{id}_A \otimes m_{2,1})f_{2,1}\big)\\ &\quad = -((a \otimes b)m_{2,1} \otimes c)f_{2,1} + (a \otimes (b \otimes c)m_{2,1})f_{2,1}\\ &\quad = -(axb \otimes c)f_{2,1} + (a \otimes xbc)f_{2,1}\\ &\quad = -abcx + abcx\\ &\quad = 0.\end{aligned}$$

For $k = 4$ we have to show that

$$0 = (f_{2,1} \otimes f_{2,1}) m_{2,1}.$$

Again, this equation holds since $f^{-1}_{2,1} = 0$. Finally, for $k \geq 5$ both sides of the equation are zero.

Now consider the identity id_{TA}. By construction, we have $Vf = \mathrm{id}_A = V\,\mathrm{id}_{TA}$, hence $\bar{V}[f] = [Vf] = [V\,\mathrm{id}_{TA}] = \bar{V}[\mathrm{id}_{TA}]$.

Assume that $[f] = [\mathrm{id}_{TA}]$, i.e. assume that f and id_{TA} are coderivation homotopic. Then there is an (f, id_{TA})-coderivation $h\colon TA \to TA$ of degree -1 such that $f - \mathrm{id}_{TA} = hm + mh$. By Lemma 37, such a coderivation is uniquely determined by its componentsa $h_{k,1}\colon A^{\otimes k} \to A$.

For degree reasons, $h_{k,1} = 0$ for $k \geq 2$, as a non-zero element of $A^{\otimes k}$ has degree $\ell \leq -k$, but $h_{k,1}$ sends it to something in A of degree $\ell - 1$. But A has only non-zero elements in degrees -1 and -2. So from $f - \mathrm{id}_{TA} = hm + mh$ we can conclude that

$$f_{2,1} = (f - \mathrm{id}_{TA})_{2,1} = (hm + mh)_{2,1} = h_{2,2} m_{2,1} + m_{2,1} h_{1,1}.$$

By Lemma 37 we have $h_{2,2} = f_{1,1} \otimes h_{1,1} + h_{1,1} \otimes \mathrm{id}_A$. But since $h^0_{1,1} = 0$, it follows that $h_{2,2} m_{2,1} = 0$. So we have $f_{2,1} = m_{2,1} h_{1,1}$.

$$\begin{array}{ccc} & & A^{-1} \otimes A^{-1} \\ & \swarrow^{f_{2,1}} & \downarrow{\scriptstyle m_{2,1}} \\ A^{-2} & \xleftarrow{h_{1,1}} & A^{-1} \end{array}$$

Restricted to $A^{-1} \otimes A^{-1}$, the map $f_{2,1}\colon A^{-1} \otimes A^{-1} \to A^{-2}$ is surjective, hence has a two-dimensional image. However, $m_{2,1}\colon A^{-1} \otimes A^{-1} \to A^{-1}$ has image in $xK[x]/(x^2)$, i.e. its image is one-dimensional. This gives a *contradiction*. □

3.3 Localisation

In this section, we show that the (coderivation) homotopy category $\underline{\mathsf{dtCoalg}}$ is the localisation of $\mathsf{dtCoalg}$ at the set of homotopy equivalences, cf. Theorem 92 below.

3.3.1 A tensor product

We construct a tensor product of a differential $\mathbf{Z}$-graded algebra and a differential $\mathcal{Z}$-graded tensor coalgebra, cf. Definition 29. Via the Bar construction differential graded tensor coalgebras correspond to A_∞-algebras. For classical A_∞-algebras, i.e. in the case when the grading category is $\mathbf{Z}$, general tensor products of A_∞-algebras have been constructed in [SU04] and [Amo12].

More precisely, for a differential graded tensor coalgebra TB, i.e. an object in $\mathsf{dtCoalg}$, we construct a functor

$$- \boxtimes TB\colon \quad \mathsf{dgAlg}_{\mathbf{Z}} \longrightarrow \mathsf{dtCoalg},$$

cf. Proposition 86 below.

Recall that *graded* means $\mathcal{Z}$-graded over a grading category $\mathcal{Z}$.

Definition 82 An A_∞-algebra $(A, (\mathsf{m}_k)_{k\geq 1})$ is called a *differential graded algebra* if $\mathsf{m}_k = 0$ for $k \geq 3$.

We abbreviate $A = (A, \mu, \delta) := (A, (\mathsf{m}_k)_{k\geq 1})$ where $\mu = \mathsf{m}_2$ is the *multiplication* and $\delta = \mathsf{m}_1$ is the *differential* of the differential graded algebra A.

The Stasheff equations for A reduce to the following three equations that hold in the differential graded algebra A.

- $(\mu \otimes \mathrm{id}_A)\mu = (\mathrm{id}_A \otimes \mu)\mu$ (Associativity)
- $\delta\delta = 0$
- $\mu\delta = (\mathrm{id}_A \otimes \delta + \delta \otimes \mathrm{id}_A)\mu$ (Leibniz rule)

We often write $ab := (a \otimes b)\mu$ for $a \otimes b \in (A \otimes A)^z$ in some degree $z \in \mathrm{Mor}(\mathcal{Z})$. Note that using this notation the Leibniz rule reads $(ab)\delta = a(b\delta) + (-1)^{\lfloor b \rfloor}(a\delta)b$.

Let $A = (A, \mu, \delta)$ and $B = (B, \mu, \delta)$ be differential graded algebras. A *morphism of differential graded algebras* $f\colon A \to B$ is a graded linear map of degree 0 such that $f\mu = \mu(f \otimes f)$ and $f\delta = \delta f$ hold.

We obtain the category dgAlg of differential graded algebras, with composition as in grMod. We write $\mathsf{dgAlg}_{\mathcal{Z}}$ if we want to make the grading category $\mathcal{Z}$ explicit.

Lemma 83 *For a* $\mathbf{Z}$*-graded module* M *let* $M^{\uparrow\mathcal{Z}}$ *be the graded module that is at* $z \in \mathrm{Mor}(\mathcal{Z})$ *given by*

$$(M^{\uparrow\mathcal{Z}})^z := \begin{cases} M^{\lfloor z \rfloor} & \text{if } z = \mathrm{id}_x[\lfloor z \rfloor] \text{ for some } x \in \mathrm{Ob}(\mathcal{Z}) \\ 0 & \text{else.} \end{cases}$$

For a $\mathbf{Z}$*-graded linear map* $f\colon M \to N$ *of degree* $p \in \mathbf{Z}$ *let* $f^{\uparrow\mathcal{Z}}\colon M^{\uparrow\mathcal{Z}} \to N^{\uparrow\mathcal{Z}}$ *be the graded linear map of degree* p *that is given at* $z \in \mathrm{Mor}(\mathcal{Z})$ *by*

$$(f^{\uparrow\mathcal{Z}})^z := \begin{cases} f^{\lfloor z \rfloor} & \text{if } z = \mathrm{id}_x[\lfloor z \rfloor] \text{ for some } x \in \mathrm{Ob}(\mathcal{Z}) \\ 0 & \text{else.} \end{cases}$$

Then the following defines a functor.

$$\begin{array}{rcl} \mathsf{grMod}_{\mathbf{Z}} & \longrightarrow & \mathsf{grMod}_{\mathcal{Z}} \\ M & \longmapsto & M^{\uparrow\mathcal{Z}} \\ (f\colon M \to N) & \longmapsto & (f^{\uparrow\mathcal{Z}}\colon M^{\uparrow\mathcal{Z}} \to N^{\uparrow\mathcal{Z}}) \end{array}$$

Proof. Let M be a $\mathbf{Z}$-graded module and let $z \in \mathrm{Mor}(\mathcal{Z})$. If $z = \mathrm{id}_x[\lfloor z \rfloor]$ for some $x \in \mathrm{Ob}(\mathcal{Z})$, we have $(M^{\uparrow\mathcal{Z}})^z = M^{\lfloor z \rfloor}$ and thus

$$(\mathrm{id}_M^{\uparrow\mathcal{Z}})^z = \mathrm{id}_M^{\lfloor z \rfloor} = \mathrm{id}_{M^{\lfloor z \rfloor}} = \mathrm{id}_{(M^{\uparrow\mathcal{Z}})^z} = \mathrm{id}^z_{M^{\uparrow\mathcal{Z}}}\,.$$

If z is not of this form, we have $(M^{\uparrow\mathcal{Z}})^z = 0$ and thus

$$(\mathrm{id}_M^{\uparrow\mathcal{Z}})^z = 0 = \mathrm{id}^z_{M^{\uparrow\mathcal{Z}}}\,.$$

We conclude that $\mathrm{id}_M^{\uparrow\mathcal{Z}} = \mathrm{id}_{M^{\uparrow\mathcal{Z}}}$ holds.

Let $f\colon L \to M$ be a $\mathbf{Z}$-graded linear map of degree $p \in \mathbf{Z}$ and let $g\colon M \to N$ be $\mathbf{Z}$-graded linear map of degree $q \in \mathbf{Z}$. Let $z \in \operatorname{Mor}(\mathcal{Z})$. If $z = \mathrm{id}_x[\lfloor z \rfloor]$ for some $x \in \operatorname{Ob}(\mathcal{Z})$, note that $z[p] = \mathrm{id}_x[\lfloor z \rfloor + p] = \mathrm{id}_x[\lfloor z[p] \rfloor]$ and thus

$$((fg)^{\uparrow\mathcal{Z}})^z = (fg)^{\lfloor z \rfloor} = f^{\lfloor z \rfloor} g^{\lfloor z \rfloor + p} = f^{\lfloor z \rfloor} g^{\lfloor z[p] \rfloor} = (f^{\uparrow\mathcal{Z}})^z (g^{\uparrow\mathcal{Z}})^{z[p]} = (f^{\uparrow\mathcal{Z}} g^{\uparrow\mathcal{Z}})^z.$$

If z is not of this form, then also $z[p]$ is not of this form. Thus we have

$$((fg)^{\uparrow\mathcal{Z}})^z = 0 = (f^{\uparrow\mathcal{Z}})^z (g^{\uparrow\mathcal{Z}})^{z[p]} = (f^{\uparrow\mathcal{Z}} g^{\uparrow\mathcal{Z}})^z.$$

We conclude that $(fg)^{\uparrow\mathcal{Z}} = f^{\uparrow\mathcal{Z}} g^{\uparrow\mathcal{Z}}$ holds. □

Lemma 84 *Let $A = (A, \mu, \delta)$ be a differential $\mathbf{Z}$-graded algebra, i.e. an object in $\mathsf{dgAlg}_{\mathbf{Z}}$. Let $TB = (TB, \Delta, m)$ be a differential graded tensor coalgebra, i.e. an object $\mathsf{dtCoalg} = \mathsf{dtCoalg}_{\mathcal{Z}}$. Let $(T(A^{\uparrow\mathcal{Z}} \otimes B), \Delta)$ be the graded tensor coalgebra over $A^{\uparrow\mathcal{Z}} \otimes B$. Consider the coderivation $\mathfrak{m}\colon T(A^{\uparrow\mathcal{Z}} \otimes B) \to T(A^{\uparrow\mathcal{Z}} \otimes B)$ of degree 1 with*

$$\mathfrak{m}_{1,1} = \delta^{\uparrow\mathcal{Z}} \otimes \mathrm{id}_B + \mathrm{id}_{A^{\uparrow\mathcal{Z}}} \otimes m_{1,1}$$

and

$$\begin{aligned} \mathfrak{m}_{k,1}\colon \quad & (A^{\uparrow\mathcal{Z}} \otimes B)^{\otimes k} \longrightarrow A^{\uparrow\mathcal{Z}} \otimes B \\ \mathfrak{m}_{k,1}^z\colon \quad & \textstyle\bigotimes_{i=1}^k a_i \otimes b_i \longmapsto (-1)^{\sum_{1\le i<j\le k} \lfloor b_i \rfloor \lfloor a_j \rfloor} a_1 \cdots a_k \otimes (b_1 \otimes \ldots \otimes b_k) m_{k,1} \end{aligned}$$

for $k \geq 2$, cf. Lemma 22.(2).
Then $A \boxtimes TB := (T(A^{\uparrow\mathcal{Z}} \otimes B), \Delta, \mathfrak{m})$ is a differential graded coalgebra.

Proof. By Lemma 24.(1) it suffices to show that

$$0 = \sum_{\substack{r+s+t=k \\ r,t \geq 0,\, s \geq 1}} (\mathrm{id}^{\otimes r} \otimes \mathfrak{m}_{s,1} \otimes \mathrm{id}^{\otimes t}) \mathfrak{m}_{r+1+t,1}$$

holds for $k \geq 1$. We write $\mathrm{id} := \mathrm{id}_{A^{\uparrow\mathcal{Z}} \otimes B}$.
Consider the case $k = 1$ first. We obtain using Lemma 83

$$\begin{aligned} \mathfrak{m}_{1,1}\mathfrak{m}_{1,1} &= (\delta^{\uparrow\mathcal{Z}} \otimes \mathrm{id}_B + \mathrm{id}_{A^{\uparrow\mathcal{Z}}} \otimes m_{1,1})(\delta^{\uparrow\mathcal{Z}} \otimes \mathrm{id}_B + \mathrm{id}_{A^{\uparrow\mathcal{Z}}} \otimes m_{1,1}) \\ &= \delta^{\uparrow\mathcal{Z}}\delta^{\uparrow\mathcal{Z}} \otimes \mathrm{id}_B + \delta^{\uparrow\mathcal{Z}} \otimes m_{1,1} - \delta^{\uparrow\mathcal{Z}} \otimes m_{1,1} + \mathrm{id}_{A^{\uparrow\mathcal{Z}}} \otimes m_{1,1}m_{1,1} \\ &= (\delta\delta)^{\uparrow\mathcal{Z}} \otimes \mathrm{id}_B + \mathrm{id}_{A^{\uparrow\mathcal{Z}}} \otimes m_{1,1}m_{1,1} \\ &= 0. \end{aligned}$$

Now let $k \geq 2$. We first separate the summands that contain a factor $\mathfrak{m}_{1,1}$.

$$\begin{aligned} \sum_{\substack{r+s+t=k \\ r,t \geq 0,\, s \geq 1}} (\mathrm{id}^{\otimes r} \otimes \mathfrak{m}_{s,1} \otimes \mathrm{id}^{\otimes t}) \mathfrak{m}_{r+1+t,1} &= \left(\sum_{\substack{r+1+t=k \\ r,t \geq 0}} (\mathrm{id}^{\otimes r} \otimes \mathfrak{m}_{1,1} \otimes \mathrm{id}^{\otimes t}) \mathfrak{m}_{k,1} \right) + \mathfrak{m}_{k,1}\mathfrak{m}_{1,1} \\ &\quad + \left(\sum_{\substack{r+s+t=k \\ r,t \geq 0;\; k-1 \geq s \geq 2}} (\mathrm{id}^{\otimes r} \otimes \mathfrak{m}_{s,1} \otimes \mathrm{id}^{\otimes t}) \mathfrak{m}_{r+1+t,1} \right) \end{aligned}$$

Now let $z \in \mathrm{Mor}(\mathcal{Z})$ and let $\bigotimes_{i=1}^{k} a_i \otimes b_i \in \big((A^{\uparrow \mathcal{Z}} \otimes B)^{\otimes k}\big)^z$. We consider the summands that contain a factor $\mathfrak{m}_{1,1}$ first.

$$\begin{aligned}
&(a_1 \otimes b_1 \otimes \ldots \otimes a_k \otimes b_k)\mathfrak{m}_{k,1}\mathfrak{m}_{1,1} \\
&\quad = (-1)^{\sum_{1\le i<j\le k} \lfloor b_i \rfloor \lfloor a_j \rfloor} (a_1 \cdots a_k \otimes (b_1 \otimes \ldots \otimes b_k) m_{k,1})(\delta^{\uparrow \mathcal{Z}} \otimes \mathrm{id}_B + \mathrm{id}_{A^{\uparrow \mathcal{Z}}} \otimes m_{1,1}) \\
&\quad = -(-1)^{\left(\sum_{1\le i<j\le k} \lfloor b_i \rfloor \lfloor a_j \rfloor\right) + \left(\sum_{i=1}^{k} \lfloor b_i \rfloor\right)} (a_1 \cdots a_k)\delta \otimes (b_1 \otimes \ldots \otimes b_k) m_{k,1} \\
&\qquad + (-1)^{\sum_{1\le i<j\le k} \lfloor b_i \rfloor \lfloor a_j \rfloor} a_1 \cdots a_k \otimes (b_1 \otimes \ldots \otimes b_k) m_{k,1} m_{1,1}
\end{aligned}$$

Moreover, we have the following summand for $r, t \geq 0$ with $r + 1 + t = k$.

$$\begin{aligned}
&(a_1 \otimes b_1 \otimes \ldots \otimes a_k \otimes b_k)(\mathrm{id}^{\otimes r} \otimes \mathfrak{m}_{1,1} \otimes \mathrm{id}^{\otimes t})\mathfrak{m}_{k,1} \\
&\quad = (-1)^{\sum_{i=r+2}^{k} \lfloor a_i \rfloor + \lfloor b_i \rfloor} \Big(\bigotimes_{i=1}^{r} (a_i \otimes b_i) \otimes (a_{r+1} \otimes b_{r+1})\mathfrak{m}_{1,1} \otimes \bigotimes_{i=r+2}^{k} (a_i \otimes b_i) \Big) \mathfrak{m}_{k,1} \\
&\quad = (-1)^{\lfloor b_{r+1} \rfloor + \left(\sum_{i=r+2}^{k} \lfloor a_i \rfloor + \lfloor b_i \rfloor\right)} \Big(\bigotimes_{i=1}^{r} (a_i \otimes b_i) \otimes a_{r+1}\delta \otimes b_{r+1} \otimes \bigotimes_{i=r+2}^{k} (a_i \otimes b_i) \Big) \mathfrak{m}_{k,1} \\
&\qquad + (-1)^{\sum_{i=r+2}^{k} \lfloor a_i \rfloor + \lfloor b_i \rfloor} \Big(\bigotimes_{i=1}^{r} (a_i \otimes b_i) \otimes a_{r+1} \otimes b_{r+1} m_{1,1} \otimes \bigotimes_{i=r+2}^{k} (a_i \otimes b_i) \Big) \mathfrak{m}_{k,1} \\
&\quad = (-1)^{\lfloor b_{r+1} \rfloor + \left(\sum_{i=r+2}^{k} \lfloor a_i \rfloor + \lfloor b_i \rfloor\right) + \left(\sum_{1\le i<j\le k} \lfloor b_i \rfloor \lfloor a_j \rfloor\right) + \left(\sum_{i=1}^{r} \lfloor b_i \rfloor\right)} \\
&\qquad\quad \cdot (a_1 \cdots a_r (a_{r+1}\delta) a_{r+2} \cdots a_k) \otimes (b_1 \otimes \ldots \otimes b_k) m_{k,1} \\
&\qquad + (-1)^{\left(\sum_{i=r+2}^{k} \lfloor a_i \rfloor + \lfloor b_i \rfloor\right) + \left(\sum_{1\le i<j\le k} \lfloor b_i \rfloor \lfloor a_j \rfloor\right) + \left(\sum_{i=r+2}^{k} \lfloor a_i \rfloor\right)} \\
&\qquad\quad \cdot (a_1 \cdots a_k) \otimes (b_1 \otimes \ldots \otimes b_r \otimes (b_{r+1}) m_{1,1} \otimes b_{r+2} \otimes \ldots \otimes b_k) m_{k,1} \\
&\quad = (-1)^{\left(\sum_{1\le i<j\le k} \lfloor b_i \rfloor \lfloor a_j \rfloor\right) + \left(\sum_{i=1}^{k} \lfloor b_i \rfloor\right) + \left(\sum_{i=r+2}^{k} \lfloor a_i \rfloor\right)} \\
&\qquad\quad \cdot (a_1 \cdots a_r (a_{r+1}\delta) a_{r+2} \cdots a_k) \otimes (b_1 \otimes \ldots \otimes b_k) m_{k,1} \\
&\qquad + (-1)^{\sum_{1\le i<j\le k} \lfloor b_i \rfloor \lfloor a_j \rfloor} \\
&\qquad\quad \cdot (a_1 \cdots a_k) \otimes (b_1 \otimes \ldots \otimes b_k)(\mathrm{id}_B^{\otimes r} \otimes m_{1,1} \otimes \mathrm{id}_B^{\otimes t}) m_{k,1}
\end{aligned}$$

Finally, we have the summands that do not contain an $\mathfrak{m}_{1,1}$, for $r, t \geq 0$ and $k - 1 \geq s \geq 2$ with $r + s + t = k$. Note that in this case $r + 1 + t \geq 2$.

$$\begin{aligned}
&(a_1 \otimes b_1 \otimes \ldots \otimes a_k \otimes b_k)(\mathrm{id}^{\otimes r} \otimes \mathfrak{m}_{s,1} \otimes \mathrm{id}^{\otimes t})\mathfrak{m}_{r+1+t,1} \\
&\quad = (-1)^{\sum_{i=r+s+1}^{k} \lfloor a_i \rfloor + \lfloor b_i \rfloor} \Big(\bigotimes_{i=1}^{r} (a_i \otimes b_i) \otimes \Big(\bigotimes_{i=r+1}^{r+s} (a_i \otimes b_i) \Big) \mathfrak{m}_{s,1} \otimes \bigotimes_{i=r+s+1}^{k} (a_i \otimes b_i) \Big) \mathfrak{m}_{r+1+t,1} \\
&\quad = (-1)^{\left(\sum_{i=r+s+1}^{k} \lfloor a_i \rfloor + \lfloor b_i \rfloor\right) + \left(\sum_{r+1\le i<j\le r+s} \lfloor b_i \rfloor \lfloor a_j \rfloor\right)} \\
&\qquad\quad \cdot \Big(\bigotimes_{i=1}^{r} (a_i \otimes b_i) \otimes a_{r+1} \cdots a_{r+s} \otimes (b_{r+1} \otimes \ldots \otimes b_{r+s}) m_{s,1} \otimes \bigotimes_{i=r+s+1}^{k} (a_i \otimes b_i) \Big) \mathfrak{m}_{r+1+t,1} \\
&\quad = (-1)^{\left(\sum_{i=r+s+1}^{k} \lfloor a_i \rfloor + \lfloor b_i \rfloor\right) + \left(\sum_{r+1\le i<j\le r+s} \lfloor b_i \rfloor \lfloor a_j \rfloor\right)} \\
&\qquad\quad \cdot (-1)^{\left(\sum_{\substack{1\le i\le r \\ i<j\le k}} \lfloor b_i \rfloor \lfloor a_j \rfloor\right) + \left(\sum_{\substack{r+1\le i\le r+s \\ r+s+1\le j\le k}} \lfloor b_i \rfloor \lfloor a_j \rfloor\right) + \left(\sum_{i=r+s+1}^{k} \lfloor a_i \rfloor\right) + \left(\sum_{r+s+1\le i<j\le k} \lfloor b_i \rfloor \lfloor a_j \rfloor\right)} \\
&\qquad\quad \cdot (a_1 \cdots a_k) \otimes (b_1 \otimes \ldots \otimes b_r \otimes (b_{r+1} \otimes \ldots \otimes b_{r+s}) m_{s,1} \otimes b_{r+s+1} \otimes \ldots \otimes b_k) m_{r+1+t,1}
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{\sum_{1\le i<j\le k}\lfloor b_i\rfloor\lfloor a_j\rfloor} \\
&\quad\cdot (a_1\cdots a_k)\otimes(b_1\otimes\ldots\otimes b_k)(\mathrm{id}_B^{\otimes r}\otimes m_{s,1}\otimes \mathrm{id}_B^{\otimes t})m_{r+1+t,1}
\end{aligned}
$$

Claim: The following equation holds for $k \geq 1$.

$$
\sum_{\substack{r+1+t=k\\ r,t\geq 0}} (-1)^{\sum_{i=r+2}^{k}\lfloor a_i\rfloor}(a_1\cdots a_r(a_{r+1}\delta)a_{r+2}\cdots a_k) = (a_1\cdots a_k)\delta.
$$

We prove this claim by induction on k. For $k=1$ both sides equal $a_1\delta$. Now assume that the equation holds for some $k\geq 1$. We have using the inductive hypothesis and the Leibniz rule for the differential $\mathbf{Z}$-graded algebra A

$$
\begin{aligned}
&\sum_{\substack{r+1+t=k+1\\ r,t\geq 0}} (-1)^{\sum_{i=r+2}^{k+1}\lfloor a_i\rfloor}(a_1\cdots a_r(a_{r+1}\delta)a_{r+2}\cdots a_{k+1}) \\
&= (-1)^{\lfloor a_{k+1}\rfloor}\left(\sum_{\substack{r+1+t=k\\ r,t\geq 0}} (-1)^{\sum_{i=r+2}^{k}\lfloor a_i\rfloor}(a_1\cdots a_r(a_{r+1}\delta)a_{r+2}\cdots a_k)\right)a_{k+1} + (a_1\cdots a_k(a_{k+1}\delta)) \\
&= (-1)^{\lfloor a_{k+1}\rfloor}((a_1\cdots a_k)\delta)a_{k+1} + (a_1\cdots a_k(a_{k+1}\delta)) \\
&= (a_1\cdots a_{k+1})\delta.
\end{aligned}
$$

This proves the *claim*.

Using this claim, the previous calculations and using Lemma 24.(1) for the differential graded coalgebra (TB,Δ,m) we obtain

$$
\begin{aligned}
&\sum_{\substack{r+s+t=k\\ r,t\geq 0,\, s\geq 1}} (a_1\otimes b_1\otimes\ldots\otimes a_k\otimes b_k)(\mathrm{id}^{\otimes r}\otimes \mathfrak{m}_{s,1}\otimes \mathrm{id}^{\otimes t})\mathfrak{m}_{r+1+t,1} \\
&\quad= \sum_{\substack{r+1+t=k\\ r,t\geq 0}} (a_1\otimes b_1\otimes\ldots\otimes a_k\otimes b_k)(\mathrm{id}^{\otimes r}\otimes \mathfrak{m}_{1,1}\otimes \mathrm{id}^{\otimes t})\mathfrak{m}_{k,1} \\
&\quad+ (a_1\otimes b_1\otimes\ldots\otimes a_k\otimes b_k)\mathfrak{m}_{k,1}\mathfrak{m}_{1,1} \\
&\quad+ \sum_{\substack{r+s+t=k\\ r,t\geq 0;\ k-1\geq s\geq 2}} (a_1\otimes b_1\otimes\ldots\otimes a_k\otimes b_k)(\mathrm{id}^{\otimes r}\otimes \mathfrak{m}_{s,1}\otimes \mathrm{id}^{\otimes t})\mathfrak{m}_{r+1+t,1} \\
&\quad= (-1)^{\sum_{1\le i<j\le k}\lfloor b_i\rfloor\lfloor a_j\rfloor} \\
&\quad\cdot\Bigg(\sum_{\substack{r+1+t=k\\ r,t\geq 0}} (-1)^{\left(\sum_{i=1}^{k}\lfloor b_i\rfloor\right)+\left(\sum_{i=r+2}^{k}\lfloor a_i\rfloor\right)}(a_1\cdots a_r(a_{r+1}\delta)a_{r+2}\cdots a_k)\otimes(b_1\otimes\ldots\otimes b_k)m_{k,1} \\
&\qquad+ \sum_{\substack{r+1+t=k\\ r,t\geq 0}} (a_1\cdots a_k)\otimes(b_1\otimes\ldots\otimes b_k)(\mathrm{id}_B^{\otimes r}\otimes m_{1,1}\otimes \mathrm{id}_B^{\otimes t})m_{k,1} \\
&\qquad- (-1)^{\sum_{i=1}^{k}\lfloor b_i\rfloor}(a_1\cdots a_k)\delta\otimes(b_1\otimes\ldots\otimes b_k)m_{k,1} \\
&\qquad+ (a_1\cdots a_k)\otimes(b_1\otimes\ldots\otimes b_k)m_{k,1}m_{1,1} \\
&\qquad+ \sum_{\substack{r+s+t=k\\ r,t\geq 0;\ k-1\geq s\geq 2}} (a_1\cdots a_k)\otimes(b_1\otimes\ldots\otimes b_k)(\mathrm{id}_B^{\otimes r}\otimes m_{s,1}\otimes \mathrm{id}_B^{\otimes t})m_{r+1+t,1}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= (-1)^{\sum_{1\le i<j\le k} \lfloor b_i \rfloor \lfloor a_j \rfloor} \\
&\cdot \Bigg((-1)^{\sum_{i=1}^{k} \lfloor b_i \rfloor} \Bigg(\sum_{\substack{r+1+t=k \\ r,t\ge 0}} (-1)^{\sum_{i=r+2}^{k} \lfloor a_i \rfloor} (a_1 \cdots a_r (a_{r+1}\delta) a_{r+2} \cdots a_k) \otimes (b_1 \otimes \ldots \otimes b_k) m_{k,1} \\
&\quad - (a_1 \cdots a_k)\delta \otimes (b_1 \otimes \ldots \otimes b_k) m_{k,1} \Bigg) \\
&\quad + \sum_{\substack{r+s+t=k \\ r,t\ge 0,\, s\ge 1}} (a_1 \cdots a_k) \otimes (b_1 \otimes \ldots \otimes b_k)(\mathrm{id}_B^{\otimes r} \otimes m_{s,1} \otimes \mathrm{id}_B^{\otimes t}) m_{r+1+t,1} \Bigg) \\
&= 0.
\end{aligned}
$$

We conclude that $A \boxtimes TB$ is a differential graded coalgebra. □

Lemma 85 *Let $A = (A, \mu, \delta)$ and $\tilde{A} = (\tilde{A}, \mu, \delta)$ be differential $\mathbf{Z}$-graded algebras, i.e. objects in $\mathsf{dgAlg}_{\mathbf{Z}}$. Let $TB = (TB, \Delta, m)$ be a differential graded tensor coalgebra. Let $f\colon A \to \tilde{A}$ be a morphism of differential $\mathbf{Z}$-graded algebras.*
Let $f \boxtimes TB\colon A \boxtimes TB \to \tilde{A} \boxtimes TB$ be the strict graded coalgebra morphism with

$$(f \boxtimes TB)_{1,1} := f^{\upharpoonright \mathcal{Z}} \otimes \mathrm{id}_B \colon A^{\upharpoonright \mathcal{Z}} \otimes B \to \tilde{A}^{\upharpoonright \mathcal{Z}} \otimes B,$$

cf. Lemma 22.(1) and Definition 69.(3).
Then $f \boxtimes TB$ is a morphism of differential graded coalgebras.

Proof. Write $\mathfrak{f} := f \boxtimes TB$. Using Lemma 24.(2) it suffices to show that for $k \ge 1$ the following equation holds.

$$\sum_{\substack{r+s+t=k \\ r,t\ge 0,\, s\ge 1}} (\mathrm{id}^{\otimes r} \otimes \mathfrak{m}_{s,1} \otimes \mathrm{id}^{\otimes t}) \mathfrak{f}_{r+1+t,1} = \sum_{\ell=1}^{k} \sum_{\substack{i_1+\ldots+i_\ell=k \\ i_1,\ldots,i_\ell \ge 1}} (\mathfrak{f}_{i_1,1} \otimes \ldots \otimes \mathfrak{f}_{i_\ell,1}) \mathfrak{m}_{k,1}$$

Since $\mathfrak{f}$ is strict, i.e. $\mathfrak{f}_{k,1} = 0$ for $k \ge 2$, it suffices to show that

$$\mathfrak{m}_{k,1}\mathfrak{f}_{1,1} = \mathfrak{f}_{1,1}^{\otimes k}\mathfrak{m}_{k,1}$$

holds for $k \ge 1$. For $k = 1$ we have

$$
\begin{aligned}
\mathfrak{m}_{1,1}\mathfrak{f}_{1,1} &= (\delta^{\upharpoonright \mathcal{Z}} \otimes \mathrm{id}_B + \mathrm{id}_{A^{\upharpoonright \mathcal{Z}}} \otimes m_{1,1})(f^{\upharpoonright \mathcal{Z}} \otimes \mathrm{id}_B) \\
&= (\delta f)^{\upharpoonright \mathcal{Z}} \otimes \mathrm{id}_B + f^{\upharpoonright \mathcal{Z}} \otimes m_{1,1} \\
&= (f \delta)^{\upharpoonright \mathcal{Z}} \otimes \mathrm{id}_B + f^{\upharpoonright \mathcal{Z}} \otimes m_{1,1} \\
&= (f^{\upharpoonright \mathcal{Z}} \otimes \mathrm{id}_B)(\delta^{\upharpoonright \mathcal{Z}} \otimes \mathrm{id}_B + \mathrm{id}_{A^{\upharpoonright \mathcal{Z}}} \otimes m_{1,1}) \\
&= \mathfrak{f}_{1,1}\mathfrak{m}_{1,1}.
\end{aligned}
$$

For $k \ge 2$, let $z \in \mathrm{Mor}(\mathcal{Z})$ and $\bigotimes_{i=1}^{k} a_i \otimes b_i \in \big((A^{\upharpoonright \mathcal{Z}} \otimes B)^{\otimes k}\big)^{z}$. Since f is a differential $\mathbf{Z}$-graded algebra morphism, it is of degree 0 and satisfies $(a_1 \cdots a_k)f = (a_1 f)\cdots(a_k f)$. Hence we obtain

$$\begin{aligned}
&(a_1 \otimes b_1 \otimes \ldots \otimes a_k \otimes b_k)\mathfrak{m}_{k,1}\mathfrak{f}_{1,1} \\
&\qquad = (-1)^{\sum_{i=1}^{k} \lfloor b_i \rfloor \lfloor a_j \rfloor} (a_1 \cdots a_k \otimes (b_1 \otimes \ldots \otimes b_k)m_{k,1})(f^{1\mathbb{Z}} \otimes \mathrm{id}_B) \\
&\qquad = (-1)^{\sum_{i=1}^{k} \lfloor b_i \rfloor \lfloor a_j \rfloor} (a_1 \cdots a_k)f \otimes (b_1 \otimes \ldots \otimes b_k)m_{k,1} \\
&\qquad = (-1)^{\sum_{i=1}^{k} \lfloor b_i \rfloor \lfloor a_j f \rfloor} (a_1 f)\cdots(a_k f) \otimes (b_1 \otimes \ldots \otimes b_k)m_{k,1} \\
&\qquad = ((a_1 f) \otimes b_1 \otimes \ldots \otimes (a_k f) \otimes b_k)\mathfrak{m}_{k,1} \\
&\qquad = (a_1 \otimes b_1 \otimes \ldots \otimes a_k \otimes b_k)(f^{1\mathbb{Z}} \otimes \mathrm{id}_B)^{\otimes k}\mathfrak{m}_{k,1} \\
&\qquad = (a_1 \otimes b_1 \otimes \ldots \otimes a_k \otimes b_k)\mathfrak{f}_{1,1}^{\otimes k}\mathfrak{m}_{k,1}. \qquad \square
\end{aligned}$$

Proposition 86 *Let $TB = (TB, \Delta, m)$ be a differential graded tensor coalgebra. Then the following defines a functor.*

$$\begin{array}{rrcl}
- \boxtimes TB\colon & \mathsf{dgAlg}_{\mathbf{Z}} & \longrightarrow & \mathsf{dtCoalg} \\
& A & \longmapsto & A \boxtimes TB \\
& (f\colon A \to \tilde{A}) & \longmapsto & (f \boxtimes TB\colon A \boxtimes TB \to \tilde{A} \boxtimes TB)
\end{array}$$

Proof. Let A be a differential $\mathbf{Z}$-graded algebra. The object $A \boxtimes TB$ in $\mathsf{dtCoalg}$ has been constructed in Lemma 84. By Lemma 85, the morphism of differential graded coalgebras $\mathrm{id}_A \boxtimes TB\colon A \boxtimes TB \to A \boxtimes TB$ is the strict graded coalgebra morphism with

$$(\mathrm{id}_A \boxtimes TB)_{1,1} = \mathrm{id}_A^{1\mathbb{Z}} \otimes \mathrm{id}_B = \mathrm{id}_{A^{1\mathbb{Z}}} \otimes \mathrm{id}_B = \mathrm{id}_{A^{1\mathbb{Z}} \otimes B}\,.$$

Hence it is the identity on $A \boxtimes TB$, which is by construction a tensor coalgebra over the graded module $A^{1\mathbb{Z}} \otimes B$.

Now let $f\colon A \to A'$ and $g\colon A' \to A''$ be morphisms of differential $\mathbf{Z}$-graded algebras between the differential $\mathbf{Z}$-graded algebras $A = (A, \mu, \delta)$, $A' = (A', \mu, \delta)$ and $A'' = (A'', \mu, \delta)$.

Since composition of strict coalgebra morphisms is again strict, also $(f \boxtimes TB)(g \boxtimes TB)$ is a strict coalgebra morphism with

$$\begin{aligned}
\big((f \boxtimes TB)(g \boxtimes TB)\big)_{1,1} &= (f \boxtimes TB)_{1,1}(g \boxtimes TB)_{1,1} \\
&= (f^{1\mathbb{Z}} \otimes \mathrm{id}_B)(g^{1\mathbb{Z}} \otimes \mathrm{id}_B) = (fg)^{1\mathbb{Z}} \otimes \mathrm{id}_B = (fg \boxtimes TB)_{1,1}.
\end{aligned}$$

Hence $(f \boxtimes TB)(g \boxtimes TB) = fg \boxtimes TB$. $\square$

Let $\dot{R}_{\mathbf{Z}}$ be the $\mathbf{Z}$-graded module with $\dot{R}_{\mathbf{Z}}^0 = R$ and $\dot{R}_{\mathbf{Z}}^z = 0$ for $z \in \mathbf{Z} \setminus \{0\}$. That is, $\dot{R}_{\mathbf{Z}}$ is the tensor unit object in the category of $\mathbf{Z}$-graded modules, cf. Remark 8. Note that $\dot{R}_{\mathbf{Z}}$ is a differential $\mathbf{Z}$-graded algebra with multiplication given by the multiplication in R and the differential being 0.

Lemma 87 *Let $TB = (TB, \Delta, m)$ be a differential graded tensor coalgebra.*
Let $\nu_{TB}\colon \dot{R}_{\mathbf{Z}} \boxtimes TB \to TB$ be the strict graded coalgebra morphism with

$$\begin{array}{rrcl}
(\nu_{TB})_{1,1}\colon & \dot{R}_{\mathbf{Z}}^{1\mathbb{Z}} \otimes B & \longrightarrow & B \\
(\nu_{TB})_{1,1}^{z}\colon & r \otimes b & \longmapsto & rb.
\end{array}$$

Then ν_{TB} is an isomorphism of differential graded coalgebras.

We will sometimes identify $\dot{R}_{\mathbf{Z}} \boxtimes TB$ and TB along ν_{TB}.

Proof. Note that $\dot{R}_{\mathbf{Z}}^{\uparrow \mathcal{Z}} = \dot{R}$ is the tensor unit object in the category of $\mathcal{Z}$-graded modules and $(\nu_{TB})_{1,1}$ is the tensor unit isomorphism, cf. Remark 8. Using Lemma 26 we conclude that ν_{TB} is an isomorphism of graded coalgebras.
To verify that ν_{TB} is an isomorphism of differential graded coalgebras, it suffices to show that ν_{TB} is a morphism of differential graded coalgebras, cf. Remark 17. Using Lemma 24.(2) it suffices to show that for $k \geq 1$ the following equation holds.

$$\sum_{\substack{r+s+t=k \\ r,t \geq 0,\, s \geq 1}} (\mathrm{id}^{\otimes r} \otimes \mathfrak{m}_{s,1} \otimes \mathrm{id}^{\otimes t})(\nu_{TB})_{r+1+t,1} = \sum_{\ell=1}^{k} \sum_{\substack{i_1+\ldots+i_\ell=k \\ i_1,\ldots,i_\ell \geq 1}} ((\nu_{TB})_{i_1,1} \otimes \ldots \otimes (\nu_{TB})_{i_\ell,1}) m_{k,1}$$

Since ν_{TB} is strict, i.e. $(\nu_{TB})_{k,1} = 0$ for $k \geq 2$, it suffices to show that

$$\mathfrak{m}_{k,1}(\nu_{TB})_{1,1} = (\nu_{TB})_{1,1}^{\otimes k} m_{k,1}$$

holds for $k \geq 1$. Let $z \in \mathrm{Mor}(\mathcal{Z})$ and $\bigotimes_{i=1}^{k} r_i \otimes b_i \in \big((\dot{R}_{\mathbf{Z}}^{\uparrow \mathcal{Z}} \otimes B)^{\otimes k}\big)^{z}$. It suffices to consider the case when $\lfloor r_i \rfloor = 0$ for $1 \leq i \leq k$. For $k = 1$ we obtain

$$\begin{aligned}
(r_1 \otimes b_1)\mathfrak{m}_{1,1}\nu_{TB} &= (r_1 \otimes b_1)(\delta^{\uparrow \mathcal{Z}} \otimes \mathrm{id}_B + \mathrm{id}_{A^{\uparrow \mathcal{Z}}} \otimes m_{1,1})\nu_{TB} \\
&= (r_1 \otimes b_1 m_{1,1})\nu_{TB} \\
&= r_1(b_1 m_{1,1}) \\
&= (r_1 b_1) m_{1,1} \\
&= (r_1 \otimes b_1)\nu_{TB} m_{1,1}.
\end{aligned}$$

For $k \geq 2$ we obtain

$$\begin{aligned}
(r_1 \otimes b_1 \otimes \ldots \otimes r_k \otimes b_k)\mathfrak{m}_{k,1}(\nu_{TB})_{1,1} &= \big((r_1 \cdots r_k) \otimes (b_1 \otimes \ldots \otimes b_k) m_{k,1}\big)(\nu_{TB})_{1,1} \\
&= (r_1 \cdots r_k)\big((b_1 \otimes \ldots \otimes b_k) m_{k,1}\big) \\
&= \big((r_1 b_1) \otimes \ldots \otimes (r_k b_k)\big) m_{k,1} \\
&= (r_1 \otimes b_1 \otimes \ldots \otimes r_k \otimes b_k)(\nu_{TB})_{1,1}^{\otimes k} m_{k,1}.
\end{aligned}$$ □

Lemma 88 *Let $f\colon A \to \tilde{A}$ be a morphism of differential $\mathbf{Z}$-graded algebras between the differential $\mathbf{Z}$-graded algebras $A = (A, \mu, \delta)$ and $\tilde{A} = (\tilde{A}, \mu, \delta)$. Let $TB = (TB, \Delta, m)$ be a differential graded tensor coalgebra. Suppose that f is a homotopy equivalence of differential $\mathbf{Z}$-graded modules between (A, δ) and $(\tilde{A}, \delta)$.*
Then $f \boxtimes TB\colon A \boxtimes TB \to \tilde{A} \boxtimes TB$ is a homotopy equivalence in dtCoalg.

Proof. By assumption there is a morphism of differential $\mathbf{Z}$-graded modules $g\colon \tilde{A} \to A$ and $\mathbf{Z}$-graded linear maps $h\colon A \to A$ and $\tilde{h}\colon \tilde{A} \to \tilde{A}$ of degree -1 such that $\mathrm{id}_A - fg = h\delta + \delta h$ and $\mathrm{id}_{\tilde{A}} - gf = \tilde{h}\delta + \delta\tilde{h}$.
We use Theorem 79 to show that $f \boxtimes TB$ is a homotopy equivalence in dtCoalg. Using this theorem, it suffices to show that $V(f \boxtimes TB) = (f \boxtimes TB)_{1,1} = f^{\uparrow \mathcal{Z}} \otimes \mathrm{id}_B$, cf. Lemma 85 for the last equality, is a homotopy equivalence of differential graded modules between $(A^{\uparrow \mathcal{Z}} \otimes B, \mathfrak{m}_{1,1})$ and $(\tilde{A}^{\uparrow \mathcal{Z}} \otimes B, \mathfrak{m}_{1,1})$. Recall that $\mathfrak{m}_{1,1} = \delta^{\uparrow \mathcal{Z}} \otimes \mathrm{id} + \mathrm{id} \otimes m_{1,1}$, cf. Lemma 84.

Consider the graded linear map $g^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B \colon \tilde{A}^{\upharpoonleft\mathbf{Z}} \otimes B \to A^{\upharpoonleft\mathbf{Z}} \otimes B$ of degree 0. Since we have

$$\begin{aligned}
(g^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B)\mathfrak{m}_{1,1} &= (g^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B)(\delta^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B + \mathrm{id}_{A^{\upharpoonleft\mathbf{Z}}} \otimes m_{1,1}) \\
&= (g\delta)^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B + g^{\upharpoonleft\mathbf{Z}} \otimes m_{1,1} \\
&= (\delta g)^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B + g^{\upharpoonleft\mathbf{Z}} \otimes m_{1,1} \\
&= (\delta^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B + \mathrm{id}_{\tilde{A}^{\upharpoonleft\mathbf{Z}}} \otimes m_{1,1})(g^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B) \\
&= \mathfrak{m}_{1,1}(g^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B)
\end{aligned}$$

the graded linear map $g^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B$ is a morphism of differential graded modules. Now consider the graded linear maps $h^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B \colon A^{\upharpoonleft\mathbf{Z}} \otimes B \to A^{\upharpoonleft\mathbf{Z}} \otimes B$ and $\tilde{h}^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B \colon \tilde{A}^{\upharpoonleft\mathbf{Z}} \otimes B \to A^{\upharpoonleft\mathbf{Z}} \otimes B$ of degree -1. Then the following equations hold.

$$\begin{aligned}
&\mathfrak{m}_{1,1}(h^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B) + (h^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B)\mathfrak{m}_{1,1} \\
&\quad = (\delta^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B + \mathrm{id}_{A^{\upharpoonleft\mathbf{Z}}} \otimes m_{1,1})(h^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B) + (h^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B)(\delta^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B + \mathrm{id}_{A^{\upharpoonleft\mathbf{Z}}} \otimes m_{1,1}) \\
&\quad = (\delta h)^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B - h^{\upharpoonleft\mathbf{Z}} \otimes m_{1,1} + (h\delta)^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B + h^{\upharpoonleft\mathbf{Z}} \otimes m_{1,1} \\
&\quad = (\delta h + h\delta)^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B \\
&\quad = (\mathrm{id}_A - fg)^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B \\
&\quad = \mathrm{id}_{A^{\upharpoonleft\mathbf{Z}}} \otimes \mathrm{id}_B - (f^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B)(g^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B) \\
&\mathfrak{m}_{1,1}(\tilde{h}^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B) + (\tilde{h}^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B)\mathfrak{m}_{1,1} \\
&\quad = (\delta^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B + \mathrm{id}_{\tilde{A}^{\upharpoonleft\mathbf{Z}}} \otimes m_{1,1})(\tilde{h}^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B) + (\tilde{h}^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B)(\delta^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B + \mathrm{id}_{\tilde{A}^{\upharpoonleft\mathbf{Z}}} \otimes m_{1,1}) \\
&\quad = (\delta\tilde{h})^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B - \tilde{h}^{\upharpoonleft\mathbf{Z}} \otimes m_{1,1} + (\tilde{h}\delta)^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B + \tilde{h}^{\upharpoonleft\mathbf{Z}} \otimes m_{1,1} \\
&\quad = (\delta\tilde{h} + \tilde{h}\delta)^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B \\
&\quad = (\mathrm{id}_{\tilde{A}} - gf)^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B \\
&\quad = \mathrm{id}_{\tilde{A}^{\upharpoonleft\mathbf{Z}}} \otimes \mathrm{id}_B - (g^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B)(f^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B)
\end{aligned}$$

This shows that $f^{\upharpoonleft\mathbf{Z}} \otimes \mathrm{id}_B$ is a homotopy equivalence of differential graded modules. □

3.3.2 The homotopy category as a localisation

We show that two homotopic maps in $\mathsf{dtCoalg}$ fit into a certain commutative diagram, cf. Lemma 91 below. We use this diagram to prove that $\underline{\mathsf{dtCoalg}}$ is the localisation of $\mathsf{dtCoalg}$ at the set of homotopy equivalences, cf. Theorem 92 below.

In the case of A_∞-algebras over a field, this commutative diagram and the interval algebra, defined in Lemma 89 below, used in its construction can be found in [Sei08, Remark 1.11].

Lemma 89 *Consider the the $\mathbf{Z}$-graded module I with $I^1 := R$, $I^0 := R \oplus R$ and $I^z := 0$ for $z \in \mathbf{Z} \setminus \{0,1\}$. Let $\delta \colon I \to I$ be the graded linear map of degree 1 with*

$$\delta^0 := \begin{pmatrix} \mathrm{id}_R \\ -\mathrm{id}_R \end{pmatrix} : I^0 = R \oplus R \to R = I^1$$

and with $\delta^z := 0$ for $z \in \mathbf{Z} \setminus \{0\}$. Let $\mu \colon I \otimes I \to I$ be the graded linear map of degree 0 given *by*

$$\begin{array}{rrcl} \mu^0\colon & I^0 \otimes I^0 & \longrightarrow & I^0 \\ & (r_0, r_1) \otimes (s_0, s_1) & \longmapsto & (r_0 s_0, r_1 s_1) \\ \mu^1\colon & I^0 \otimes I^1 \oplus I^1 \otimes I^0 & \longrightarrow & I^1 \\ & ((r_0, r_1) \otimes t,\, \tilde{t} \otimes (\tilde{r}_0, \tilde{r}_1)) & \longmapsto & r_0 t + \tilde{t}\tilde{r}_1 \end{array}$$

and by $\mu^z = 0$ *for* $z \in \mathbf{Z} \setminus \{0, 1\}$.
Then $I = (I, \mu, \delta)$ *is a differential* $\mathbf{Z}$*-graded algebra, the* interval algebra.

Proof. Since $\delta^z \neq 0$ only if $z = 0$, we have $\delta\delta = 0$. Hence δ is a differential.
We verify associativity of the multiplication μ, i.e. we verify that $(\mathrm{id}_I \otimes \mu)^z \mu^z = (\mu \otimes \mathrm{id}_I)^z \mu^z$ holds for $z \in \mathbf{Z}$. Since $\mu^z = 0$ for $z \in \mathbf{Z} \setminus \{0, 1\}$, we only have to consider the cases $z = 0$ and $z = 1$.
For $z = 0$, note that $(I \otimes I \otimes I)^0 = I^0 \otimes I^0 \otimes I^0$. Let $(r_0, r_1) \otimes (s_0, s_1) \otimes (t_0, t_1) \in I^0 \otimes I^0 \otimes I^0$. We obtain

$$\begin{aligned} ((r_0, r_1) \otimes (s_0, s_1) \otimes (t_0, t_1))(\mathrm{id}_I \otimes \mu)\mu &= ((r_0, r_1) \otimes (s_0 t_0, s_1 t_1))\mu \\ &= (r_0 s_0 t_0, r_1 s_1 t_1) \\ ((r_0, r_1) \otimes (s_0, s_1) \otimes (t_0, t_1))(\mu \otimes \mathrm{id}_I)\mu &= ((r_0 s_0, r_1 s_1) \otimes (t_0, t_1))\mu \\ &= (r_0 s_0 t_0, r_1 s_1 t_1). \end{aligned}$$

For $z = 1$, note that $(I \otimes I \otimes I)^1 = (I^1 \otimes I^0 \otimes I^0) \oplus (I^0 \otimes I^1 \otimes I^0) \oplus (I^0 \otimes I^0 \otimes I^1)$.
Let $t \otimes (r_0, r_1) \otimes (s_0, s_1) \in I^1 \otimes I^0 \otimes I^0$. We obtain

$$\begin{aligned} (t \otimes (r_0, r_1) \otimes (s_0, s_1))(\mathrm{id}_I \otimes \mu)\mu &= (t \otimes (r_0 s_0, r_1 s_1))\mu \\ &= t r_1 s_1 \\ (t \otimes (r_0, r_1) \otimes (s_0, s_1))(\mu \otimes \mathrm{id}_I)\mu &= (t r_1 \otimes (s_0, s_1))\mu \\ &= t r_1 s_1. \end{aligned}$$

Let $(r_0, r_1) \otimes t \otimes (s_0, s_1) \in I^0 \otimes I^1 \otimes I^0$. We obtain

$$\begin{aligned} ((r_0, r_1) \otimes t) \otimes (s_0, s_1))(\mathrm{id}_I \otimes \mu)\mu &= ((r_0, r_1) \otimes t s_1)\mu \\ &= r_0 t s_1 \\ ((r_0, r_1) \otimes t \otimes (s_0, s_1))(\mu \otimes \mathrm{id}_I)\mu &= (r_0 t \otimes (s_0, s_1))\mu \\ &= r_0 t s_1. \end{aligned}$$

Let $(r_0, r_1) \otimes (s_0, s_1) \otimes t \in I^0 \otimes I^0 \otimes I^1$. We obtain

$$\begin{aligned} ((r_0, r_1) \otimes (s_0, s_1) \otimes t)(\mathrm{id}_I \otimes \mu)\mu &= ((r_0, r_1) \otimes s_0 t))\mu \\ &= r_0 s_0 t \\ ((r_0, r_1) \otimes (s_0, s_1) \otimes t)(\mu \otimes \mathrm{id}_I)\mu &= ((r_0 s_0, r_1 s_1) \otimes t))\mu \\ &= r_0 s_0 t. \end{aligned}$$

We verify the Leibniz rule, i.e. we verify that $(\mathrm{id}_I \otimes \delta + \delta \otimes \mathrm{id}_I)^z \mu^{z+1} = \mu^z \delta^z \colon (I \otimes I)^z \to I^{z+1}$ holds for $z \in \mathbf{Z}$. Since $I^z = 0$ for $z \in \mathbf{Z} \setminus \{0, 1\}$, it suffices to consider the Leibniz rule for the case $z = 0$.
Note that $(I \otimes I)^0 = I^0 \otimes I^0$. Let $(r_0, r_1) \otimes (s_0, s_1) \in I^0 \otimes I^0$. We obtain

$$\begin{aligned}
((r_0, r_1) \otimes (s_0, s_1))(\mathrm{id}_I \otimes \delta + \delta \otimes \mathrm{id}_I)\mu &= ((r_0, r_1) \otimes (s_0 - s_1) + (r_0 - r_1) \otimes (s_0, s_1))\mu \\
&= r_0(s_0 - s_1) + (r_0 - r_1)s_1 \\
&= r_0 s_0 - r_1 s_1. \\
((r_0, r_1) \otimes (s_0, s_1))\mu\delta &= (r_0 s_0, r_1 s_1)\delta \\
&= r_0 s_0 - r_1 s_1.
\end{aligned}$$

□

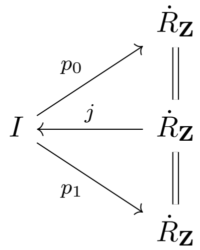
Lemma 90 *Define the* $\mathbf{Z}$*-graded linear maps of degree* 0

$$\begin{array}{lccc}
p_0\colon & I & \longrightarrow & \dot{R}_{\mathbf{Z}} \\
p_0^0\colon & (r_0, r_1) & \longmapsto & r_0
\end{array}
\qquad
\begin{array}{lccc}
p_1\colon & I & \longrightarrow & \dot{R}_{\mathbf{Z}} \\
p_1^0\colon & (r_0, r_1) & \longmapsto & r_1,
\end{array}$$

where $p_0^z = 0$ *and* $p_1^z = 0$ *for* $z \in \mathbf{Z} \setminus \{0\}$.
Moreover, define the $\mathbf{Z}$*-graded linear map of degree* 0

$$\begin{array}{lccc}
j\colon & \dot{R}_{\mathbf{Z}} & \longrightarrow & I \\
j^0\colon & r & \longmapsto & (r, r),
\end{array}$$

where $j^z = 0$ *for* $z \in \mathbf{Z} \setminus \{0\}$.
Then p_0*,* p_1 *and* j *are morphisms of differential* $\mathbf{Z}$*-graded algebras and the following diagram commutes.*



Moreover, p_0*,* p_1 *and* j *are homotopy equivalences of differential* $\mathbf{Z}$*-graded modules between* (I, δ) *and* $(\dot{R}_{\mathbf{Z}}, 0)$*.*

Proof. Both p_0 and p_1 are morphisms of differential $\mathbf{Z}$-graded modules, as $\dot{R}_{\mathbf{Z}}^z = 0$ for $z \in \{0\}$ and $I^{-1} = 0$. To show that p_0 is a morphism of differential $\mathbf{Z}$-graded algebras, it suffices to show that $(p_0^0 \otimes p_0^0)\mu^0 = \mu^0 p_0^0$. But for $(r_0, r_1) \otimes (s_0, s_1) \in I^0 \otimes I^0 = (I \otimes I)^0$ we have

$$((r_0, r_1) \otimes (s_0, s_1))(p_0 \otimes p_0)\mu = r_0 s_0 = (r_0 s_0, r_1 s_1)p_0 = ((r_0, r_1) \otimes (s_0, s_1))\mu p_0.$$

A similar argument shows that p_1 is a morphism of differential $\mathbf{Z}$-graded algebras.
To show that j is a morphism of differential $\mathbf{Z}$-graded modules, we have to show that $j^0 \delta^0 = 0$. But for $r \in R = \dot{R}_{\mathbf{Z}}^0$ we obtain

$$r j^0 \delta^0 = (r, r)\delta^0 = r - r = 0.$$

Hence j is a morphism of differential $\mathbf{Z}$-graded modules. To show that j is a morphism of differential $\mathbf{Z}$-graded algebras, we have to show that $(j^0 \otimes j^0)\mu^0 = \mu^0 j^0$. But for $r, s \in R$ we have

$$(r \otimes s)(j \otimes j)\mu = ((r,r) \otimes (s,s))\mu = (rs, rs) = (rs)j = (r \otimes s)\mu j.$$

Hence j is a morphism of differential $\mathbf{Z}$-graded algebras.
For the equation $jp_0 = \mathrm{id}_{\dot{R}_{\mathbf{Z}}}$, it suffices to show that $j^0 p_0^0 = \mathrm{id}_R$. But for $r \in R = \dot{R}^0_{\mathbf{Z}}$ we have

$$rj^0 p_0^0 = (r,r)p_0^0 = r.$$

The same argument shows that $jp_1 = \mathrm{id}_{\dot{R}_{\mathbf{Z}}}$.
To show that p_0, p_1 and j are homotopy equivalences of differential $\mathbf{Z}$-graded modules, it suffices to show that j is a homotopy equivalence. Indeed, if j is a homotopy equivalence then the equations $jp_0 = \mathrm{id}_{\dot{R}_{\mathbf{Z}}}$ and $jp_1 = \mathrm{id}_{\dot{R}_{\mathbf{Z}}}$ imply that p_0 and p_1 are homotopy equivalences. We already know that $jp_1 = \mathrm{id}_{\dot{R}_{\mathbf{Z}}}$. So it remains to show that $p_1 j$ is homotopic to id_I. Consider the $\mathbf{Z}$-graded linear map of degree -1

$$\begin{array}{rccc} h_1\colon & I & \longrightarrow & I \\ h_1^1\colon & r & \longmapsto & (r,0), \end{array}$$

where $h_1^z = 0$ for $z \in \mathbf{Z} \setminus \{1\}$. We claim that $\mathrm{id}_I - p_0 j = \delta h_1 + h_1 \delta$. It suffices to show thst $\mathrm{id}_{I^z} - p_1^z j^z = \delta^z h_1^{z+1} + h_1^z \delta^{z-1}$ holds for $z \in \{0,1\}$.
For $z = 0$, we have to show that $\mathrm{id}_{R \oplus R} - p_1^0 j^0 = \delta^0 h_1^1$. But for $(r_0, r_1) \in R \oplus R = I^0$ we have

$$\begin{aligned} (r_0, r_1) - (r_0, r_1)p_1^0 j^0 &= (r_0, r_1) - r_1 j^0 = (r_0, r_1) - (r_1, r_1) = (r_0 - r_1, 0) \\ (r_0, r_1)\delta^0 h_1^1 &= (r_0 - r_1)h_1^1 = (r_0 - r_1, 0). \end{aligned}$$

For $z = 1$, we have to show that $\mathrm{id}_R = h_1^1 \delta^0$. But for $r \in R = I^1$ we have

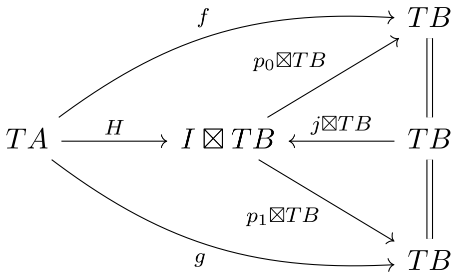
$$rh_1^1 \delta^0 = (r,0)\delta^0 = r. \qquad \square$$

Lemma 91 *Let $TA = (TA, \Delta, m)$ and $TB = (TB, \Delta, m)$ be differential graded tensor coalgebras.*
Let $f\colon TA \to TB$ and $g\colon TA \to TB$ be morphisms of differential graded coalgebras.
Let $h\colon TA \to TB$ be an (f,g)-coderivation of degree -1, cf. Definition 34. Consider the graded coalgebra morphism $H\colon TA \to I \boxtimes TB$ given by

$$\begin{array}{rccl} H_{k,1}\colon & A^{\otimes k} & \longrightarrow & I^{\uparrow \mathcal{Z}} \otimes B \\ H_{k,1}^z\colon & a_1 \otimes \ldots \otimes a_k & \longmapsto & \underbrace{(1,0)}_{\in (I^{\uparrow \mathcal{Z}})^{\mathrm{id}_x} = I^0} \otimes (a_1 \otimes \ldots \otimes a_k) f_{k,1} + \underbrace{(0,1)}_{\in (I^{\uparrow \mathcal{Z}})^{\mathrm{id}_x} = I^0} \otimes (a_1 \otimes \ldots \otimes a_k) g_{k,1} \\ & & & -(-1)^{\sum_{i=1}^k \lfloor a_i \rfloor} \cdot \underbrace{1}_{\in (I^{\uparrow \mathcal{Z}})^{\mathrm{id}_x[1]} = I^1} \otimes (a_1 \otimes \ldots \otimes a_k) h_{k,1}, \end{array}$$

for $k \geq 1$ and $z\colon x \to y$ in $\mathcal{Z}$. This defines a graded coalgebra morphism by Lemma 22.(1). Then H is a morphism of differential graded coalgebras if and only if $f - g = hm + mh$, i.e. if and only if h is a coderivation homotopy between f and g, cf. Definition 57.

Moreover, if h is a coderivation homotopy from f to g, then we have the following commutative diagram in dtCoalg.



Recall that we identify along the tensor unit isomorphism ν_{TB} *from Lemma* 87.

Proof. By Lemma 24.(2) the graded coalgebra morphism H is a morphism of differential graded coalgebras if and only if the Stasheff equation for morphisms holds for $k \geq 1$.

$$\sum_{\substack{r+s+t=k \\ r,t\geq 0,\, s\geq 1}} (\mathrm{id}_A^{\otimes r} \otimes m_{s,1} \otimes \mathrm{id}_A^{\otimes t}) H_{r+1+t,1} = \sum_{\ell=1}^{k} \sum_{\substack{i_1+\ldots+i_\ell=k \\ i_1,\ldots,i_\ell \geq 1}} (H_{i_1,1} \otimes \ldots \otimes H_{i_\ell,1}) \mathfrak{m}_{\ell,1}$$

Here $\mathfrak{m}$ denotes the differential on $I \boxtimes TB$, cf. Lemma 84.
Let $z \in \mathrm{Mor}(\mathcal{Z})$ and let $a_1 \otimes \ldots \otimes a_k \in (A^{\otimes k})^z$. We obtain for a summand in the left-hand side

$$\begin{aligned}
&(a_1 \otimes \ldots \otimes a_k)(\mathrm{id}_A^{\otimes r} \otimes m_{s,1} \otimes \mathrm{id}_A^{\otimes t}) H_{r+1+t,1} \\
&\ = (-1)^{\sum_{i=r+s+1}^{k} \lfloor a_i \rfloor} (a_1 \otimes \ldots \otimes a_r \otimes (a_{r+1} \otimes \ldots \otimes a_{r+s}) m_{s,1} \otimes a_{r+s+1} \otimes \ldots \otimes a_k) H_{r+1+t,1} \\
&\ = (-1)^{\sum_{i=r+s+1}^{k} \lfloor a_i \rfloor} \\
&\quad \cdot \Big((1,0) \otimes (a_1 \otimes \ldots \otimes a_r \otimes (a_{r+1} \otimes \ldots \otimes a_{r+s}) m_{s,1} \otimes a_{r+s+1} \otimes \ldots \otimes a_k) f_{r+1+t,1} \\
&\quad + (0,1) \otimes (a_1 \otimes \ldots \otimes a_r \otimes (a_{r+1} \otimes \ldots \otimes a_{r+s}) m_{s,1} \otimes a_{r+s+1} \otimes \ldots \otimes a_k) g_{r+1+t,1} \\
&\quad + (-1)^{\sum_{i=1}^{k} \lfloor a_i \rfloor} 1 \otimes (a_1 \otimes \ldots \otimes a_r \otimes (a_{r+1} \otimes \ldots \otimes a_{r+s}) m_{s,1} \otimes a_{r+s+1} \otimes \ldots \otimes a_k) h_{r+1+t,1} \Big) \\
&\ = (1,0) \otimes (a_1 \otimes \ldots \otimes a_k)(\mathrm{id}_A^{\otimes r} \otimes m_{s,1} \otimes \mathrm{id}_A^{\otimes t}) f_{r+1+t,1} \\
&\quad + (0,1) \otimes (a_1 \otimes \ldots \otimes a_k)(\mathrm{id}_A^{\otimes r} \otimes m_{s,1} \otimes \mathrm{id}_A^{\otimes t}) g_{r+1+t,1} \\
&\quad + (-1)^{\sum_{i=1}^{k} \lfloor a_i \rfloor} 1 \otimes (a_1 \otimes \ldots \otimes a_k)(\mathrm{id}_A^{\otimes r} \otimes m_{s,1} \otimes \mathrm{id}_A^{\otimes t}) h_{r+1+t,1}.
\end{aligned}$$

On the other hand, we obtain for a summand in the right-hand side

$$\begin{aligned}
&(a_1 \otimes \ldots \otimes a_k)(H_{i_1,1} \otimes \ldots \otimes H_{i_\ell,1}) \mathfrak{m}_{\ell,1} \\
&\ = \Big(\bigotimes_{u=1}^{\ell} (a_{i_1+\ldots+i_{u-1}+1} \otimes \ldots \otimes a_{i_1+\ldots+i_u}) H_{i_u,1} \Big) \mathfrak{m}_{\ell,1} \\
&\ = \Big(\bigotimes_{u=1}^{\ell} \Big(\underbrace{(1,0)}_{=:\alpha_{0,u}} \otimes \underbrace{(a_{i_1+\ldots+i_{u-1}+1} \otimes \ldots \otimes a_{i_1+\ldots+i_u}) f_{i_u,1}}_{=:\beta_{0,u}} \\
&\qquad + \underbrace{(0,1)}_{=:\alpha_{1,u}} \otimes \underbrace{(a_{i_1+\ldots+i_{u-1}+1} \otimes \ldots \otimes a_{i_1+\ldots+i_u}) g_{i_u,1}}_{=:\beta_{1,u}}
\end{aligned}$$

$$- \underbrace{1}_{=:\alpha_{2,u}} \otimes \underbrace{(-1)^{\sum_{j=i_1+\ldots+i_{u-1}+1}^{i_1+\ldots+i_u}\lfloor a_j\rfloor}(a_{i_1+\ldots+i_{u-1}+1}\otimes\ldots\otimes a_{i_1+\ldots+i_u})h_{i_u,1}\Big)}_{=:\beta_{2,u}}\Big)\mathfrak{m}_{\ell,1}$$

$$= \sum_{(v_1,\ldots,v_\ell)\in\{0,1,2\}^{\times\ell}}\Big(\bigotimes_{u=1}^{\ell}(\alpha_{v_u,u}\otimes\beta_{v_u,u})\Big)\mathfrak{m}_{\ell,1}. \qquad (*)$$

We continue with the case $\ell = 1$ first.

$$\begin{aligned}
(*) &= \sum_{v_1\in\{0,1,2\}}(\alpha_{v_1,1}\otimes\beta_{v_1,1})(\delta\otimes\mathrm{id}_B+\mathrm{id}_{I\uparrow\mathbb{Z}}\otimes m_{1,1})\\
&= (-1)^{\sum_{i=1}^{k}\lfloor a_i\rfloor}(1,0)\delta\otimes(a_1\otimes\ldots\otimes a_k)f_{k,1}+(-1)^{\sum_{i=1}^{k}\lfloor a_i\rfloor}(0,1)\delta\otimes(a_1\otimes\ldots\otimes a_k)g_{k,1}\\
&\quad+(-1)^{\sum_{i=1}^{k}\lfloor a_i\rfloor}1\delta\otimes(-1)^{\sum_{i=1}^{k}\lfloor a_i\rfloor}(a_1\otimes\ldots\otimes a_k)h_{k,1}\\
&\quad+(1,0)\otimes(a_1\otimes\ldots\otimes a_k)f_{k,1}m_{1,1}+(0,1)\otimes(a_1\otimes\ldots\otimes a_k)g_{k,1}m_{1,1}\\
&\quad-1\otimes(-1)^{\sum_{i=1}^{k}\lfloor a_i\rfloor}(a_1\otimes\ldots\otimes a_k)h_{k,1}m_{1,1}\\
&= (-1)^{\sum_{i=1}^{k}\lfloor a_i\rfloor}1\otimes(a_1\otimes\ldots\otimes a_k)f_{k,1}-(-1)^{\sum_{i=1}^{k}\lfloor a_i\rfloor}1\otimes(a_1\otimes\ldots\otimes a_k)g_{k,1}\\
&\quad+(1,0)\otimes(a_1\otimes\ldots\otimes a_k)f_{k,1}m_{1,1}+(0,1)\otimes(a_1\otimes\ldots\otimes a_k)g_{k,1}m_{1,1}\\
&\quad-(-1)^{\sum_{i=1}^{k}\lfloor a_i\rfloor}1\otimes(a_1\otimes\ldots\otimes a_k)h_{k,1}m_{1,1}
\end{aligned}$$

Now we consider the case $\ell \geq 2$ in $(*)$.

$$(*) = \sum_{(v_1,\ldots,v_\ell)\in\{0,1,2\}^{\times\ell}}(-1)^{\sum_{1\leq i<j\leq\ell}\lfloor\beta_{v_i,i}\rfloor\lfloor\alpha_{v_j,j}\rfloor}(\alpha_{v_1,1}\cdots\alpha_{v_\ell,\ell})\otimes(\beta_{v_1,1}\otimes\ldots\otimes\beta_{v_\ell,\ell})m_{\ell,1}$$

Note that the product in the first tensor factor is non-zero only if the tuple $(v_1,\ldots,v_\ell)$ equals $(0,\ldots,0)$, $(1,\ldots,1)$ or is of the form $(0,\ldots,0,2,1,\ldots,1)$. In these cases, we have

$$\begin{aligned}
\alpha_{0,1}\cdots\alpha_{0,\ell} &= (1,0)\cdots(1,0)=(1,0)\\
\alpha_{1,1}\cdots\alpha_{1,\ell} &= (0,1)\cdots(0,1)=(0,1)\\
\alpha_{0,1}\cdots\alpha_{0,r}\alpha_{2,r+1}\alpha_{1,r+2}\cdots\alpha_{1,\ell} &= (1,0)\cdots(1,0)\cdot1\cdot(0,1)\cdots(0,1)=1,
\end{aligned}$$

where $0\leq r\leq\ell-1$. Thus we obtain, using that $\lfloor\alpha_{0,u}\rfloor=\lfloor\alpha_{1,u}\rfloor=0$ and $\lfloor\alpha_{2,u}\rfloor=1$ for $1\leq u\leq l$,

$$\begin{aligned}
(*) &= (1,0)\otimes(\beta_{0,1}\otimes\ldots\otimes\beta_{0,\ell})m_{\ell,1}+(0,1)\otimes(\beta_{1,1}\otimes\ldots\otimes\beta_{1,\ell})m_{\ell,1}\\
&\quad+\sum_{\substack{r+1+t=\ell\\ r,t\geq0}}(-1)^{\sum_{j=1}^{r}\lfloor\beta_{0,j}\rfloor}1\otimes(\beta_{0,1}\otimes\ldots\otimes\beta_{0,r}\otimes\beta_{2,r+1}\otimes\beta_{1,r+2}\otimes\ldots\otimes\beta_{1,\ell})m_{\ell,1}\\
&= (1,0)\otimes\Big(\bigotimes_{u=1}^{\ell}(a_{i_1+\ldots+i_{u-1}+1}\otimes\ldots\otimes a_{i_1+\ldots+i_u})f_{i_u,1}\Big)m_{\ell,1}\\
&\quad+(0,1)\otimes\Big(\bigotimes_{u=1}^{\ell}(a_{i_1+\ldots+i_{u-1}+1}\otimes\ldots\otimes a_{i_1+\ldots+i_u})g_{i_u,1}\Big)m_{\ell,1}\\
&\quad+\sum_{\substack{r+1+t=\ell\\ r,t\geq0}}(-1)^{1+\sum_{j=1}^{i_1+\ldots+i_r}\lfloor a_j\rfloor}(-1)^{\sum_{j=i_1+\ldots+i_r+1}^{i_1+\ldots+i_{r+1}}\lfloor a_j\rfloor}
\end{aligned}$$

$$1 \otimes \Bigg(\Big(\bigotimes_{u=1}^{r}(a_{i_1+\ldots+i_{u-1}+1} \otimes \ldots \otimes a_{i_1+\ldots+i_u})f_{i_u,1}\Big) \otimes (a_{i_1+\ldots+i_r+1} \otimes \ldots \otimes a_{i_1+\ldots+i_{r+1}})h_{i_{r+1},1}$$

$$\otimes \Big(\bigotimes_{u=r+2}^{\ell}(a_{i_1+\ldots+i_{u-1}+1} \otimes \ldots \otimes a_{i_1+\ldots+i_u})g_{i_u,1}\Big)\Bigg)m_{\ell,1}$$

$$= (1,0) \otimes (a_1 \otimes \ldots \otimes a_k)(f_{i_1,1} \otimes \ldots \otimes f_{i_\ell,1})m_{\ell,1}$$

$$+ (0,1) \otimes (a_1 \otimes \ldots \otimes a_k)(g_{i_1,1} \otimes \ldots \otimes g_{i_\ell,1})m_{\ell,1}$$

$$- \sum_{\substack{r+1+t=\ell \\ r,t\geq 0}} (-1)^{\sum_{j=1}^{k}\lfloor a_j \rfloor}$$

$$1 \otimes (a_1 \otimes \ldots \otimes a_k)(f_{i_1,1} \otimes \ldots \otimes f_{i_r,1} \otimes h_{i_{r+1},1} \otimes g_{i_{r+2},1} \otimes \ldots \otimes g_{i_\ell,1})m_{\ell,1}$$

To summarise, we obtain for the left-hand side of the Stasheff equation for morphisms

$$\sum_{\substack{r+s+t=k \\ r,t\geq 0,\, s\geq 1}} (a_1 \otimes \ldots \otimes a_k)(\mathrm{id}_A^{\otimes r} \otimes m_{s,1} \otimes \mathrm{id}_A^{\otimes t})H_{r+1+t,1}$$

$$= \sum_{\substack{r+s+t=k \\ r,t\geq 0,\, s\geq 1}} (1,0) \otimes (a_1 \otimes \ldots \otimes a_k)(\mathrm{id}_A^{\otimes r} \otimes m_{s,1} \otimes \mathrm{id}_A^{\otimes t})f_{r+1+t,1}$$

$$+ \sum_{\substack{r+s+t=k \\ r,t\geq 0,\, s\geq 1}} (0,1) \otimes (a_1 \otimes \ldots \otimes a_k)(\mathrm{id}_A^{\otimes r} \otimes m_{s,1} \otimes \mathrm{id}_A^{\otimes t})g_{r+1+t,1}$$

$$= \sum_{\substack{r+s+t=k \\ r,t\geq 0,\, s\geq 1}} (-1)^{\sum_{i=1}^{k}\lfloor a_i \rfloor} 1 \otimes (a_1 \otimes \ldots \otimes a_k)(\mathrm{id}_A^{\otimes r} \otimes m_{s,1} \otimes \mathrm{id}_A^{\otimes t})h_{r+1+t,1}.$$

On the other hand, we obtain for the right-hand side of the Stasheff equation for morphism

$$\sum_{\ell=1}^{k} \sum_{\substack{i_1+\ldots+i_\ell=k \\ i_1,\ldots,i_\ell\geq 1}} (a_1 \otimes \ldots \otimes a_k)(H_{i_1,1} \otimes \ldots \otimes H_{i_\ell,1})\mathfrak{m}_{\ell,1}$$

$$= (-1)^{\sum_{i=1}^{k}\lfloor a_i \rfloor} 1 \otimes (a_1 \otimes \ldots \otimes a_k)f_{k,1} - (-1)^{\sum_{i=1}^{k}\lfloor a_i \rfloor} 1 \otimes (a_1 \otimes \ldots \otimes a_k)g_{k,1}$$

$$+ \sum_{\ell=1}^{k} \sum_{\substack{i_1+\ldots+i_\ell=k \\ i_1,\ldots,i_\ell\geq 1}} (1,0) \otimes (a_1 \otimes \ldots \otimes a_k)(f_{i_1,1} \otimes \ldots \otimes f_{i_\ell,1})m_{\ell,1}$$

$$+ \sum_{\ell=1}^{k} \sum_{\substack{i_1+\ldots+i_\ell=k \\ i_1,\ldots,i_\ell\geq 1}} (0,1) \otimes (a_1 \otimes \ldots \otimes a_k)(g_{i_1,1} \otimes \ldots \otimes g_{i_\ell,1})m_{\ell,1}$$

$$- \sum_{\ell=1}^{k} \sum_{\substack{i_1+\ldots+i_\ell=k \\ i_1,\ldots,i_\ell\geq 1}} \sum_{\substack{r+1+t=\ell \\ r,t\geq 0}} (-1)^{\sum_{j=1}^{k}\lfloor a_j \rfloor}$$

$$1 \otimes (a_1 \otimes \ldots \otimes a_k)(f_{i_1,1} \otimes \ldots \otimes f_{i_r,1} \otimes h_{i_{r+1},1} \otimes g_{i_{r+2},1} \otimes \ldots \otimes g_{i_\ell,1})m_{\ell,1}$$

Since f and g are morphisms of differential graded coalgebras, the Stasheff equation for morphisms holds for them, cf. Lemma 24.(2). Hence the Stasheff equation for morphisms for

H holds if and only if the following equation holds for $k \geq 1$.

$$\sum_{\substack{r+s+t=k \\ r,t\geq 0,\, s\geq 1}} (-1)^{\sum_{i=1}^{k}\lfloor a_i \rfloor} 1 \otimes (a_1 \otimes \ldots \otimes a_k)(\mathrm{id}_A^{\otimes r} \otimes m_{s,1} \otimes \mathrm{id}_A^{\otimes t}) h_{r+1+t,1}$$
$$= (-1)^{\sum_{i=1}^{k}\lfloor a_i \rfloor} 1 \otimes (a_1 \otimes \ldots \otimes a_k) f_{k,1} - (-1)^{\sum_{i=1}^{k}\lfloor a_i \rfloor} 1 \otimes (a_1 \otimes \ldots \otimes a_k) g_{k,1}$$
$$- \sum_{\ell=1}^{k} \sum_{\substack{i_1+\ldots+i_\ell=k \\ i_1,\ldots,i_\ell\geq 1}} \sum_{\substack{r+1+t=\ell \\ r,t\geq 0}} (-1)^{\sum_{j=1}^{k}\lfloor a_j \rfloor}$$
$$1 \otimes (a_1 \otimes \ldots \otimes a_k)(f_{i_1,1} \otimes \ldots \otimes f_{i_r,1} \otimes h_{i_{r+1},1} \otimes g_{i_{r+2},1} \otimes \ldots \otimes g_{i_\ell,1}) m_{\ell,1}$$

But this equation holds if and only if

$$f_{k,1} - g_{k,1} = \sum_{\substack{r+s+t=k \\ r,t\geq 0,\, s\geq 1}} (\mathrm{id}_A^{\otimes r} \otimes m_{s,1} \otimes \mathrm{id}_A^{\otimes t}) h_{r+1+t,1}$$
$$+ \sum_{\ell=1}^{k} \sum_{\substack{i_1+\ldots+i_\ell=k \\ i_1,\ldots,i_\ell\geq 1}} \sum_{\substack{r+1+t=\ell \\ r,t\geq 0}} (f_{i_1,1} \otimes \ldots \otimes f_{i_r,1} \otimes h_{i_{r+1},1} \otimes g_{i_{r+2},1} \otimes \ldots \otimes g_{i_\ell,1}) m_{\ell,1}$$

holds for $k \geq 1$. Consider the sums on the right-hand side. The first one equals using Lemma 22.(2)

$$\sum_{\substack{r+s+t=k \\ r,t\geq 0,\, s\geq 1}} (\mathrm{id}_A^{\otimes r} \otimes m_{s,1} \otimes \mathrm{id}_A^{\otimes t}) h_{r+1+t,1} = \sum_{\ell=1}^{k} \sum_{\substack{r+s+t=k \\ r+1+t=\ell \\ r,t\geq 0,\, s\geq 1}} (\mathrm{id}_A^{\otimes r} \otimes m_{s,1} \otimes \mathrm{id}_A^{\otimes t}) h_{\ell,1} = \sum_{\ell=1}^{k} m_{k,\ell} h_{\ell,1}.$$

The second one equals using Lemma 22.(1), Remark 32 and Lemma 37

$$\sum_{\ell=1}^{k} \sum_{\substack{i_1+\ldots+i_\ell=k \\ i_1,\ldots,i_\ell\geq 1}} \sum_{\substack{r+1+t=\ell \\ r,t\geq 0}} (f_{i_1,1} \otimes \ldots \otimes f_{i_r,1} \otimes h_{i_{r+1},1} \otimes g_{i_{r+2},1} \otimes \ldots \otimes g_{i_\ell,1}) m_{\ell,1}$$
$$= \sum_{\ell=1}^{k} \sum_{\substack{u+s+v=k \\ r+1+t=\ell \\ r,t,u,v\geq 0,\, s\geq 1}} \sum_{\substack{i_1+\ldots+i_r=u \\ i_1,\ldots,i_r\geq 1}} \sum_{\substack{i_{r+2}+\ldots+i_\ell=v \\ i_{r+2},\ldots,i_\ell\geq 1}} (f_{i_1,1} \otimes \ldots \otimes f_{i_r,1} \otimes h_{s,1} \otimes g_{i_{r+2},1} \otimes \ldots \otimes g_{i_\ell,1}) m_{\ell,1}$$
$$= \sum_{\ell=1}^{k} \sum_{\substack{u+s+v=k \\ r+1+t=\ell \\ r,t,u,v\geq 0,\, s\geq 1}} (\hat{f}_{u,r} \otimes h_{s,1} \otimes \hat{g}_{v,t}) m_{\ell,1}$$
$$= \sum_{\ell=1}^{k} h_{k,\ell} m_{\ell,1}.$$

Hence the Stasheff equation for morphisms for H holds if and only if the following equation holds for $k \geq 1$.

$$f_{k,1} - g_{k,1} = (hm)_{k,1} + (mh)_{k,1}.$$

By Remark 35 and Remark 59, both $f - g$ and $mh + hm$ are (f, g)-coderivations of degree 0. By Corollary 38 two (f, g)-coderivations are equal if and only if their $(k, 1)$-components are equal for $k \geq 1$. So we conclude that H is a morphism of differential graded coalgebras if and only if $f - g = hm + mh$ holds.

It remains to verify the asserted commutativites. The equations $(j \boxtimes TB)(p_0 \boxtimes TB) = \mathrm{id}_{TB}$ and $(j \boxtimes TB)(p_1 \boxtimes TB) = \mathrm{id}_{TB}$ follow from the previous Lemma 90.

It remains to verify that $H(p_0 \boxtimes TB) = f$ and $H(p_1 \boxtimes TB) = g$ hold. As these are equations of graded coalgebra morphisms, it suffices to show that

$$(H(p_0 \boxtimes TB))_{k,1} = f_{k,1} \quad \text{and} \quad (H(p_1 \boxtimes TB))_{k,1} = g_{k,1}$$

hold for $k \geq 1$, cf. Lemma 22.(1). However, in Lemma 85 we constructed $p_0 \boxtimes TB$ and $p_1 \boxtimes TB$ as strict morphisms of graded coalgebras. Hence we have

$$(H(p_0 \boxtimes TB))_{k,1} = \sum_{\ell=1}^{k} H_{k,\ell}(p_0 \boxtimes TB)_{\ell,1} = H_{k,1}(p_0 \boxtimes TB)_{1,1}$$

and similarly $(H(p_1 \boxtimes TB))_{k,1} = H_{k,1}(p_1 \boxtimes TB)_{1,1}$.

Let $z \in \mathrm{Mor}(\mathcal{Z})$ and $a_1 \otimes \ldots \otimes a_k \in (A^{\otimes k})^z$. Recall that we identify along the tensor unit isomorphism ν_{TB} from Lemma 87. We obtain

$$\begin{aligned}
&(a_1 \otimes \ldots \otimes a_k)H_{k,1}(p_0 \boxtimes TB)_{1,1} \\
&\quad = ((1,0) \otimes (a_1 \otimes \ldots \otimes a_k)f_{k,1})(p_0^{\upharpoonright \mathcal{Z}} \otimes \mathrm{id}_B) + ((0,1) \otimes (a_1 \otimes \ldots \otimes a_k)g_{k,1})(p_0^{\upharpoonright \mathcal{Z}} \otimes \mathrm{id}_B) \\
&\qquad - (-1)^{\sum_{i=1}^{k} \lfloor a_i \rfloor} \cdot (1 \otimes (a_1 \otimes \ldots \otimes a_k)h_{k,1})(p_0^{\upharpoonright \mathcal{Z}} \otimes \mathrm{id}_B) \\
&\quad = 1 \otimes (a_1 \otimes \ldots \otimes a_k)f_{k,1} \\
&\quad = (a_1 \otimes \ldots \otimes a_k)f_{k,1}.
\end{aligned}$$

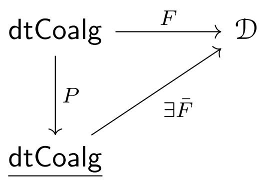
Hence $H(p_0 \boxtimes TB) = f$ holds. Similarly, we have

$$\begin{aligned}
&(a_1 \otimes \ldots \otimes a_k)H_{k,1}(p_1 \boxtimes TB)_{1,1} \\
&\quad = ((1,0) \otimes (a_1 \otimes \ldots \otimes a_k)f_{k,1})(p_1^{\upharpoonright \mathcal{Z}} \otimes \mathrm{id}_B) + ((0,1) \otimes (a_1 \otimes \ldots \otimes a_k)g_{k,1})(p_1^{\upharpoonright \mathcal{Z}} \otimes \mathrm{id}_B) \\
&\qquad - (-1)^{\sum_{i=1}^{k} \lfloor a_i \rfloor} \cdot (1 \otimes (a_1 \otimes \ldots \otimes a_k)h_{k,1})(p_1^{\upharpoonright \mathcal{Z}} \otimes \mathrm{id}_B) \\
&\quad = 1 \otimes (a_1 \otimes \ldots \otimes a_k)g_{k,1} \\
&\quad = (a_1 \otimes \ldots \otimes a_k)g_{k,1}.
\end{aligned}$$

Hence $H(p_1 \boxtimes TB) = g$ holds. □

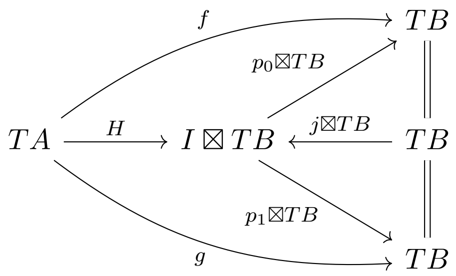
Theorem 92 *Let $\mathcal{D}$ be a category. Let $F\colon$ dtCoalg $\to \mathcal{D}$ be a functor such that for each homotopy equivalence f in dtCoalg the image Ff is an isomorphism in $\mathcal{D}$.*

Then there exists a unique functor $\bar{F}\colon$ <u>dtCoalg</u> $\to \mathcal{D}$ such that $F = \bar{F} \circ P$ holds, where $P\colon$ dtCoalg $\to$ <u>dtCoalg</u> denotes the residue class functor.

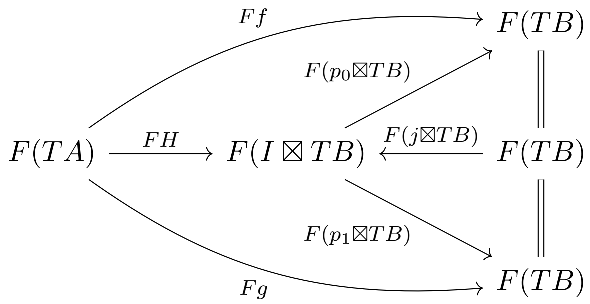


Proof. Let $f\colon TA \to TB$ and $g\colon TA \to TB$ be two morphisms in $\mathsf{dtCoalg}$ that are coderivation homotopic. Since $\underline{\mathsf{dtCoalg}}$ is defined as the factor category of $\mathsf{dtCoalg}$ modulo coderivation homotopy, it suffices to show that $Ff = Fg$ holds.

By Lemma 91, there is a differential graded coalgebra morphism $H\colon TA \to I \boxtimes TB$ such that the following diagram commutes.



By Lemma 90 both p_0 and p_1 are homotopy equivalences of differential $\mathbf{Z}$-graded modules. Thus Lemma 88 implies that $p_0 \boxtimes TB$ and $p_1 \boxtimes TB$ are homotopy equivalences in $\mathsf{dtCoalg}$. Applying the functor F to this diagram we obtain the following commutative diagram in $\mathcal{D}$.



By assumption, $F(p_0 \boxtimes TB)$ and $F(p_1 \boxtimes TB)$ are isomorphisms. Hence the equation

$$\big(F(j \boxtimes TB)\big)\big(F(p_0 \boxtimes TB)\big) = \mathrm{id}_{F(TB)} = \big(F(j \boxtimes TB)\big)\big(F(p_1 \boxtimes TB)\big)$$

implies that

$$\big(F(p_0 \boxtimes TB)\big)^{-1} = F(j \boxtimes TB) = \big(F(p_1 \boxtimes TB)\big)^{-1}.$$

So we have $F(p_0 \boxtimes TB) = F(p_1 \boxtimes TB)$. But then

$$Ff = (FH)\big(F(p_0 \boxtimes TB)\big) = (FH)\big(F(p_1 \boxtimes TB)\big) = Fg. \qquad \square$$

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Zusammenfassung

Wir konstruieren die Homotopiekategorie von A_∞-Kategorien und untersuchen Homotopieäquivalenzen. Wir arbeiten durchgehend über einem kommutativen Ring R.

Wir führen den Formalismus von Graduierungskategorien ein. Damit können wir A_∞-Kategorien als A_∞-Algebren handhaben.

Wir konstruieren den Bar-Funktor, der eine Äquivalenz zwischen der Kategorie $\mathrm{A}_\infty\text{-}\mathsf{alg}$ der A_∞-Algebren und einer vollen Teilkategorie $\mathsf{dtCoalg}$ der differentiell graduierten Coalgebren $\mathsf{dgCoalg}$ herstellt.

$$\mathrm{Bar}\colon \quad \mathrm{A}_\infty\text{-}\mathsf{alg} \xrightarrow{\sim} \mathsf{dtCoalg} \subseteq \mathsf{dgCoalg}$$

Die Kategorie $\mathsf{dtCoalg}$ enthält alle differentiell graduierten Coalgebren, deren unterliegende graduierte Coalgebra eine Tensorcoalgebra ist. Wir arbeiten durchgehend auf der Coalgebrenseite des Bar-Funktors, d.h. in $\mathsf{dtCoalg}$.

Zur Konstruktion der Homotopiekategorie führen wir verallgemeinerte (f, g)-Coderivationen ein. Wir konstruieren eine A_∞-Kategorie auf diesen Coderivationen.

Wir definieren den Begriff der Coderivationshomotopie und zeigen, dass dies eine Kongruenz auf $\mathsf{dtCoalg}$ definiert. Um Symmetrie und Transitivität dieser Relation zu zeigen, benötigen wir gewisse Korrekturterme, die von der A_∞-Kategorie auf den Coderivationen produziert werden.

Wir erhalten die Homotopiekategorie $\underline{\mathsf{dtCoalg}}$. Mit Hilfe des Bar-Funktors übersetzt sich Coderivationshomotopie zu A_∞-Homotopie und wir erhalten die Homotopiekategorie $\mathrm{A}_\infty\text{-}\underline{\mathsf{alg}}$ der A_∞-Algebren.

Nach der Konstruktion der Homotopiekategorie wollen wir Homotopieäquivalenzen charakterisieren. Dazu führen wir einen Funktor $V\colon \mathsf{dtCoalg} \to \mathsf{dgMod}$ ein, der die Tensorcoalgebra TA auf den graduierten Modul A mit eingeschränktem Differential und einen Morphismus $f\colon TA \to TB$ auf die Einschränkung $f|_A^B$ schickt. Wir zeigen, dass V einen Funktor $\bar{V}\colon \underline{\mathsf{dtCoalg}} \to \underline{\mathsf{dgMod}}$ zwischen den Homotopiekategorien induziert.

$$\begin{array}{ccccc}
\mathrm{A}_\infty\text{-}\mathsf{alg} & \xrightarrow[\sim]{\mathrm{Bar}} & \mathsf{dtCoalg} & \xrightarrow{V} & \mathsf{dgMod} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{A}_\infty\text{-}\underline{\mathsf{alg}} & \xrightarrow[\sim]{} & \underline{\mathsf{dtCoalg}} & \xrightarrow{\bar{V}} & \underline{\mathsf{dgMod}}
\end{array}$$

Als Resultat erhalten wir, dass $\bar{V}$ Isomorphismen reflektiert. In anderen Worten, ein Morphismus $f\colon TA \to TB$ ist eine Homotopieäquivalenz genau dann, wenn die Einschränkung $f|_A^B$ eine Homotopieäquivalenz in dgMod ist. Diese Charakterisierung verallgemeinert ein Resultat von Prouté.

Wir konstruieren Beispiele, die zeigen, dass $\bar{V}$ im Allgemeinen weder voll noch treu ist.

Schließlich zeigen wir, dass die Homotopiekategorie $\underline{\mathsf{dtCoalg}}$ die Lokalisierung von $\mathsf{dtCoalg}$ an den Homotopieäquivalenzen ist. Dazu zeigen wir, dass zwei coderivationshomotope Morphismen in $\mathsf{dtCoalg}$ in ein gewisses kommutatives Diagramm passen.

Hiermit versichere ich,

(1) dass ich meine Arbeit selbstständig verfasst habe,

(2) dass ich keine anderen als die angegeben Quellen benutzt habe und alle wörtlich oder sinngemäß aus anderen Werken übernommenen Aussagen als solche gekennzeichnet habe,

(3) dass die eingereichte Arbeit weder vollständig noch in wesentlichen Teilen Gegenstand eines anderen Prüfungsverfahrens gewesen ist und

(4) dass das elektronische Exemplar mit den anderen Exemplaren übereinstimmt.

Stuttgart, August 2018

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