Summer term 2024 Dr. Andreas Mihatsch
Elliptic curves and their moduli spaces

Exam dates: August 5 (1st), and September 16 (2nd). At 9 am. Location will be announced in due time.

Homework problems (due July 5)

Problem 1 (Gradings and $\mathbb{G}_m$-actions)

(a) Complete the proof of Proposition 3.14:[1] Let A be a ring together with a $\mathbb{Z}$-grading $A = \bigoplus_{i\in\mathbb{Z}} A_i$. That is, $A = \bigoplus_{i\in\mathbb{Z}} A_i$ as abelian group and $A_iA_j \subseteq A_{i+j}$. Prove that there is an action $\mu : \mathbb{G}_m \times \operatorname{Spec} A \to \operatorname{Spec} A$ with $\mu^*(a_i) = t^i \otimes a_i$ for all $a_i \in A_i$.

(b) Let k be a field and let $X = (\mathbb{A}^1_k \setminus \{0\}) \times_k \mathbb{A}^1_k$. Define a $\mathbb{G}_{m,k}$-action

$$\mu : \mathbb{G}_{m,k} \times_k X \longrightarrow X$$

such that on k-valued points we obtain

$$\mu(k) : \quad \lambda \cdot (x, y) = (\lambda x, \lambda^{-1} y).$$

Determine the corresponding $\mathbb{Z}$-grading on $\Gamma(X, \mathcal{O}_X)$.

Problem 2 (Quadratic twists over $\mathbb{Q}$)

Suppose $a, b \in \mathbb{Q}$ with $4a^3 + 27b^2 \neq 0$ and $D \in \mathbb{Q}^\times$. Consider the two elliptic curves defined by the Weierstrass equations

$$E : \; y^2 = x^3 + ax + b, \qquad E_D : \; Dy^2 = x^3 + ax + b.$$

Assume that $a, b \in \mathbb{Q}^\times$ and that D is not a square. Show that then $E \ncong E_D$.

Hint: First transform the second Weierstrass equation into simplified form. Then show that there is no substitution $x' = ux$, $y' = vy$ with $u^3 = v^2$, where $u, v \in \mathbb{Q}^\times$, that transforms the first simple equation into the second.

Further Problems

Problem 3 (Gradings and μ_n-actions)

Recall that $\mu_n = \operatorname{Spec} \mathbb{Z}[t]/(t^n - 1)$ denotes the group scheme of n-th roots of unity. Show that for any ring A there is a bijection

$$\{\mu_n\text{-actions on } \operatorname{Spec} A\} \longleftrightarrow \Big\{\text{Gradings } A = \bigoplus_{i\in\mathbb{Z}/n\mathbb{Z}} A_i\Big\}.$$

[1] For simplicity, we take $R = \mathbb{Z}$.

Problem 4 (Pullback of $\Omega^1_{X/S}$)

Let $X \to S$ be a separated morphism of schemes and let $\sigma : S \to X$ be a section. Let $\mathcal{I}$ be the ideal sheaf defining the closed subscheme $\sigma(S) \subseteq X$. Assume that X/S is smooth. Prove that there is an isomorphism

$$\mathcal{I}/\mathcal{I}^2 \xrightarrow{\sim} \sigma^*(\Omega^1_{X/S}), \quad f \longmapsto df.$$