Geometrically defined cycles on moduli spaces of curves

Johannes Schmitt

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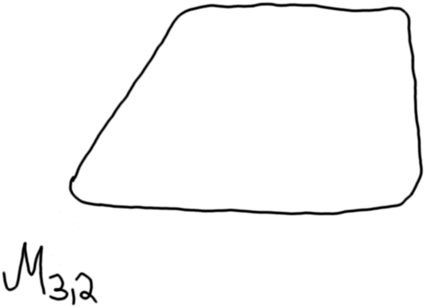

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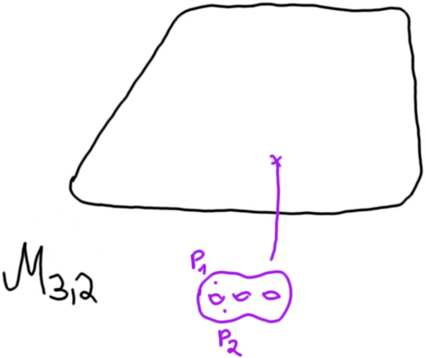

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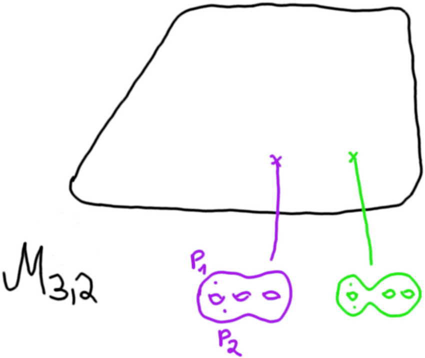

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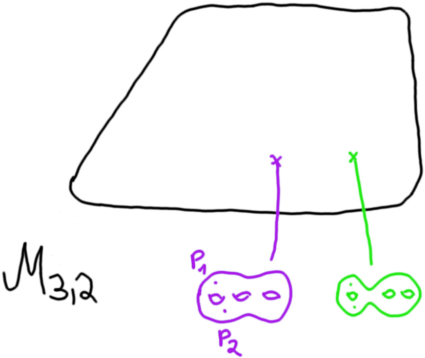

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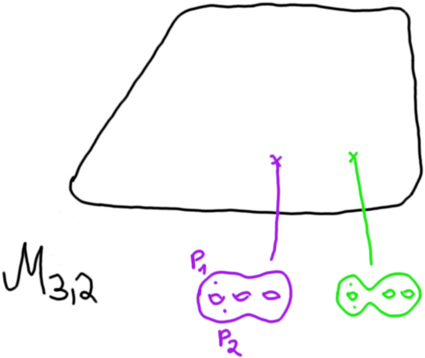

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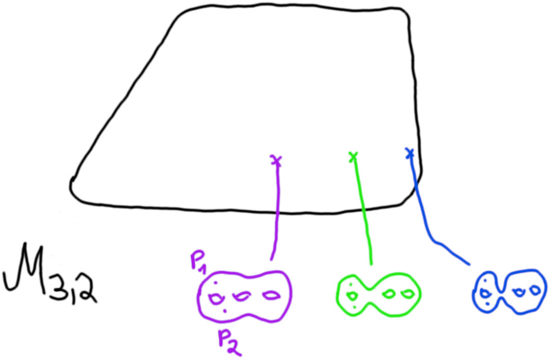

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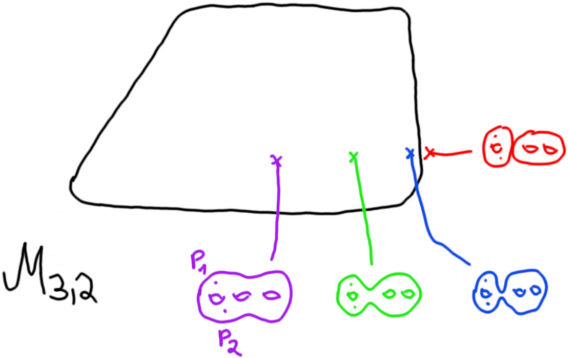

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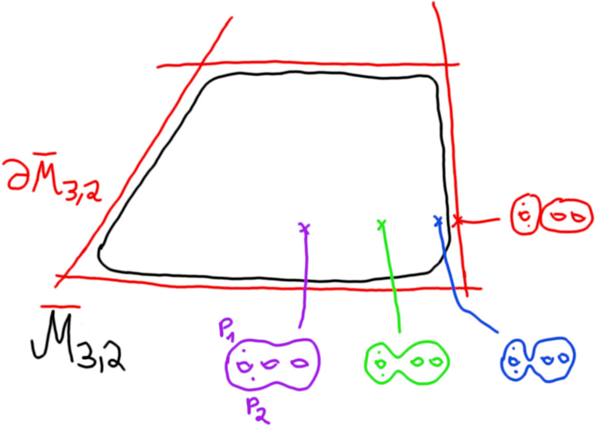

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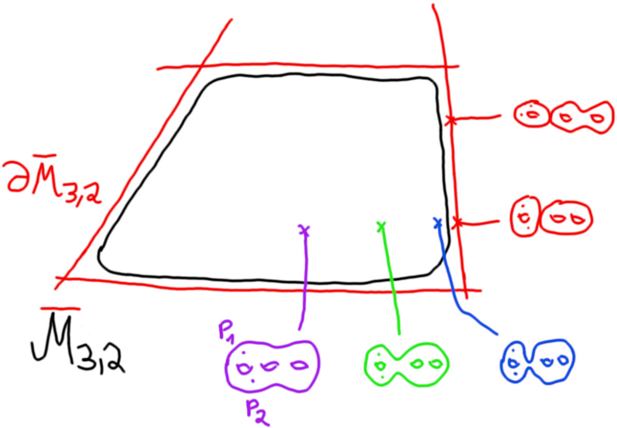

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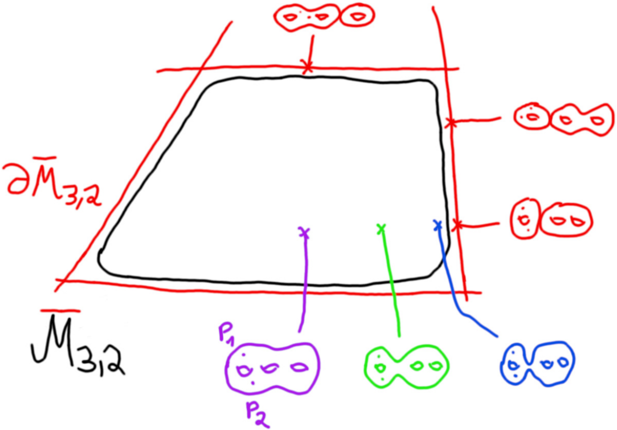

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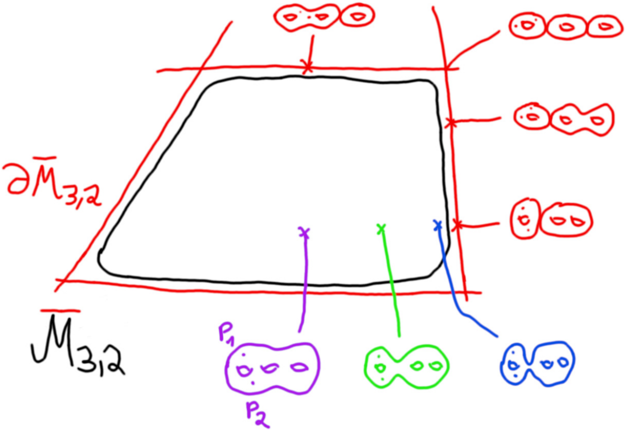

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Definition (Deligne-Mumford 1969)

Let $g, n \geq 0$ be integers (with $2g-2+n>0$).

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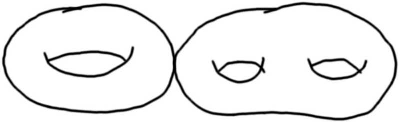

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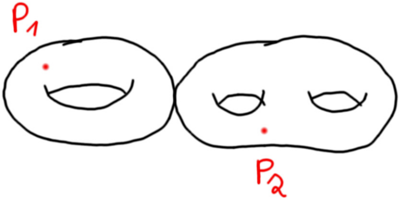

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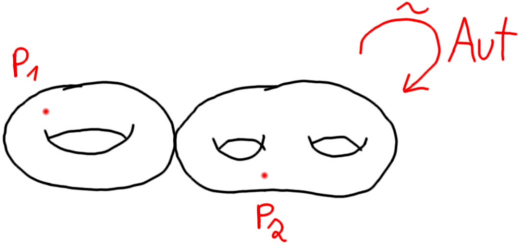

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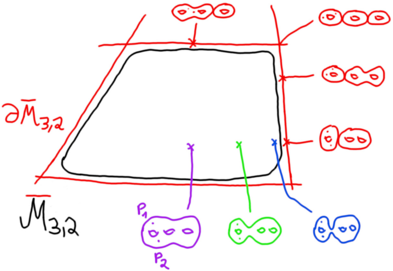
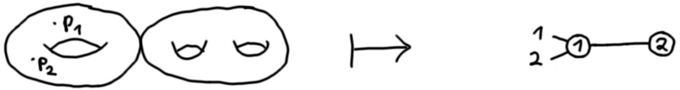


Facts

1. $\overline{\mathcal{M}}_{g,n}$ is a smooth, connected, compact space of $\mathbb{C}$-dimension $3g-3+n$.
2. The boundary $\partial\overline{\mathcal{M}}_{g,n} = \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ is a closed subset of $\mathbb{C}$-codimension 1 (normal crossing divisor), parametrized by products of smaller-dimensional spaces $\overline{\mathcal{M}}_{g_i,n_i}$.

Recursive boundary structure

To $(C, p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{g,n}$ we can associate a stable graph $\Gamma_{(C,p_1,\ldots,p_n)}$



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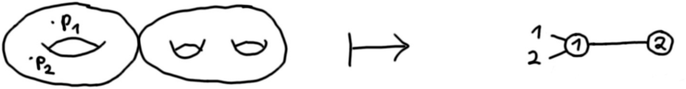
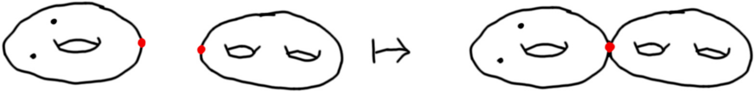


Conversely, given a stable graph Γ we have a gluing map

$$\xi_\Gamma : \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)} = \overline{\mathcal{M}}_{1,3} \times \overline{\mathcal{M}}_{2,1} \to \overline{\mathcal{M}}_{3,2}$$



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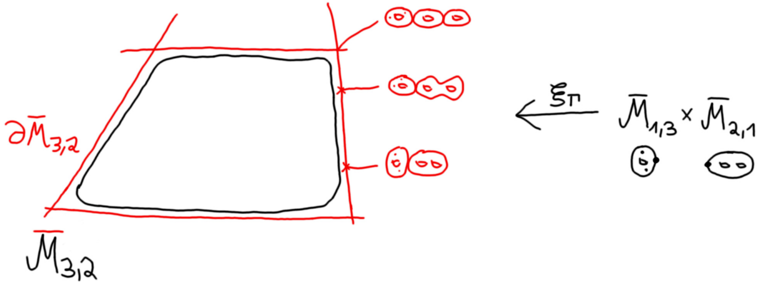

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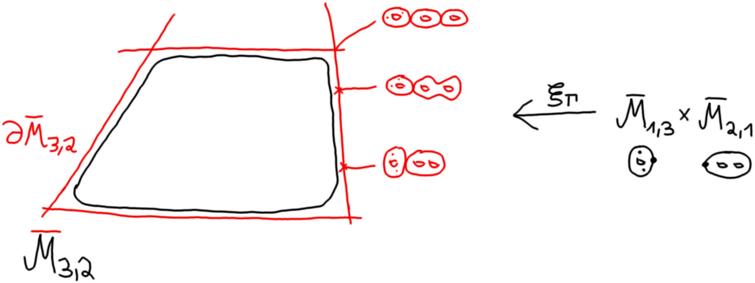

Proposition

The map ξ_Γ is finite with image equal to

$$\overline{\{(C, p_1, \ldots, p_n) : \Gamma_{(C,p_1,\ldots,p_n)} = \Gamma\}}.$$

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- $\overline{\mathcal{M}}_{g,n}$ compact space $\implies$ $H^*(\overline{\mathcal{M}}_{g,n})$ finite-dimensional $\mathbb{Q}$-algebra
- (Poincaré duality) For all $0 \leq k \leq \dim = 2(3g-3+n)$, the cup product defines a nondegenerate pairing

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- For $S \subset \overline{\mathcal{M}}_{g,n}$ a closed, algebraic subset of $\mathbb{C}$-codimension d, there exists a fundamental class

$$[S] \in H_{\dim - 2d}(\overline{\mathcal{M}}_{g,n}) \underset{\text{PD}}{\cong} H^{2d}(\overline{\mathcal{M}}_{g,n}).$$

Natural cohomology classes on $\overline{\mathcal{M}}_{g,n}$

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Definition: ψ-classes

$\mathbb{L}_i \to \overline{\mathcal{M}}_{g,n}$ complex line bundle, $\mathbb{L}_i|_{(C,p_1,\ldots,p_n)} = T^*_{p_i} C$

$$\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n}).$$

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Definition: κ-classes

Forgetful morphism

$F : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}, (C, p_1, \ldots, p_n, p_{n+1}) \mapsto (C, p_1, \ldots, p_n)$ [C smooth]

$$\kappa_a = F_* \left((\psi_{n+1})^{a+1} \right) \in H^{2a}(\overline{\mathcal{M}}_{g,n}).$$

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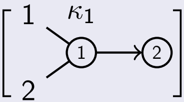

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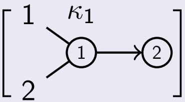

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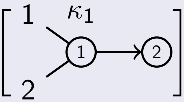

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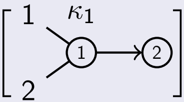

$= (\xi_\Gamma)_* (\kappa_1 \otimes \psi_h) \in RH^*(\overline{\mathcal{M}}_{3,2})$,

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Properties of the tautological ring

- explicit, finite list of generators $[\Gamma, \alpha]$ as $\mathbb{Q}$-vector space
- combinatorial description of cup product $[\Gamma, \alpha] \cdot [\Gamma', \alpha']$ (Graber-Pandharipande, 2003)
- list of many linear relations between the generators (Faber-Zagier 2000, Pandharipande-Pixton 2010, Pixton 2012, Pandharipande-Pixton-Zvonkine 2013)
- effective description of isomorphism $RH^{\dim}(\overline{\mathcal{M}}_{g,n}) \cong \mathbb{Q}$ (Witten 1991, Kontsevich 1992)

Geometrically defined cycles

Heuristic

For many algebraic-geometric properties $\mathcal{P}$ of smooth pointed curves $(C, p_1, \ldots, p_n)$ (e.g. $\mathcal{P}(C) =$ "C is hyperelliptic"):

$$S_{\mathcal{P}} = \{(C, p_1, \ldots, p_n) \in \mathcal{M}_{g,n} : \mathcal{P}(C, p_1, \ldots, p_n) \text{ is true}\} \underset{\substack{\text{closed} \\ \text{algebraic}}}{\subset} \mathcal{M}_{g,n}$$

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Goal

Decide if $[\overline{S_{\mathcal{P}}}] \in H^*(\overline{\mathcal{M}}_{g,n})$ lies in $RH^*(\overline{\mathcal{M}}_{g,n})$. If so, compute formula in terms of generators.

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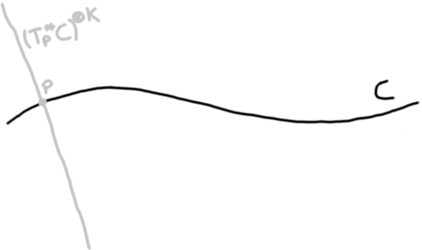

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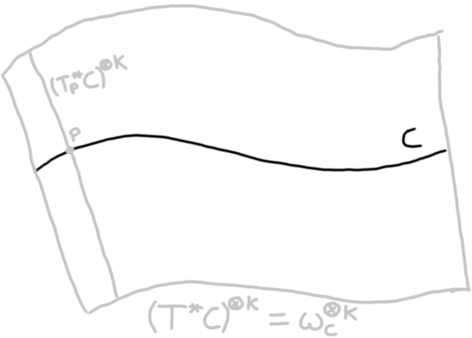

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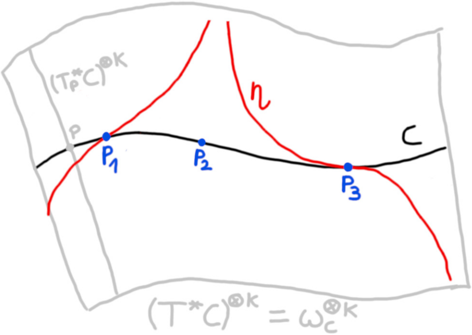

Strata of meromorphic k-differentials

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Given $g, n, k \geq 0$ and $\mu = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ with $\sum_i m_i = k(2g-2)$, let

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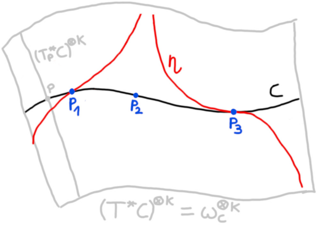

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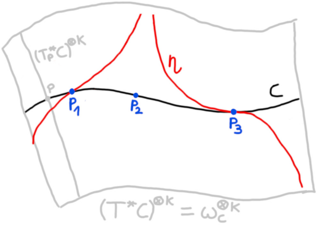

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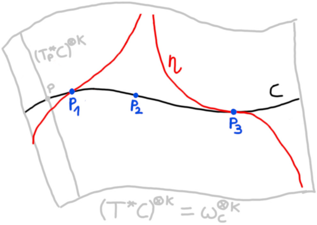
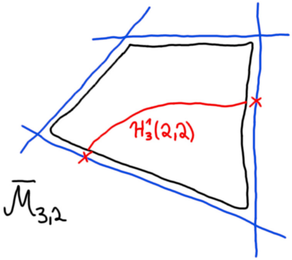
$$\mathcal{H}_g^k(\mu) = \left\{ (C, p_1, \ldots, p_n) : \begin{array}{l} \exists \text{ meromorphic } k\text{-differential } \eta \\ \text{on } C \text{ with zeros/poles at} \\ p_i \in C \text{ of orders } m_i \end{array} \right\} \subset \mathcal{M}_{g,n}$$

$$= \left\{ (C, p_1, \ldots, p_n) : \omega_C^{\otimes k} \cong \mathcal{O}_C\Big(\sum_i m_i p_i\Big) \right\} \subset \mathcal{M}_{g,n}$$



Compactifying $\mathcal{H}_g^k(\mu)$: twisted differentials



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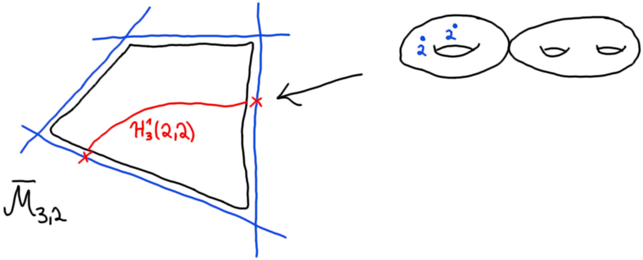

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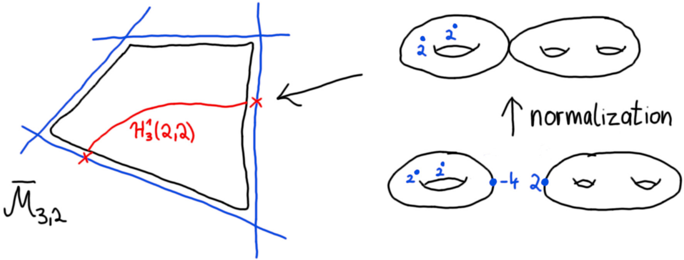

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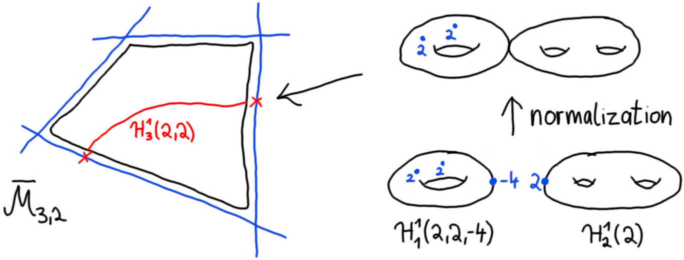

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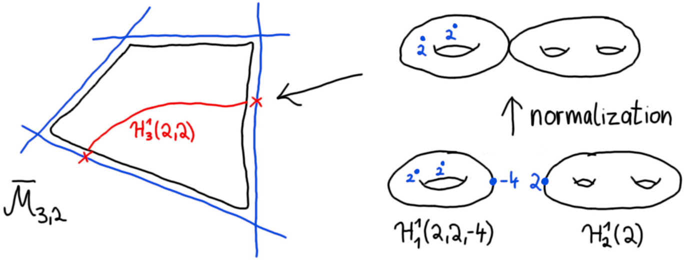

Definition (Farkas-Pandharipande 2015)

$$\widetilde{\mathcal{H}}_g^k(\mu) = \left\{ (C, p_1, \ldots, p_n) : \left(\begin{array}{l} \text{equality of line bundles on} \\ \text{partial normalization of } C \end{array} \right) \right\} \subset \overline{\mathcal{M}}_{g,n}$$

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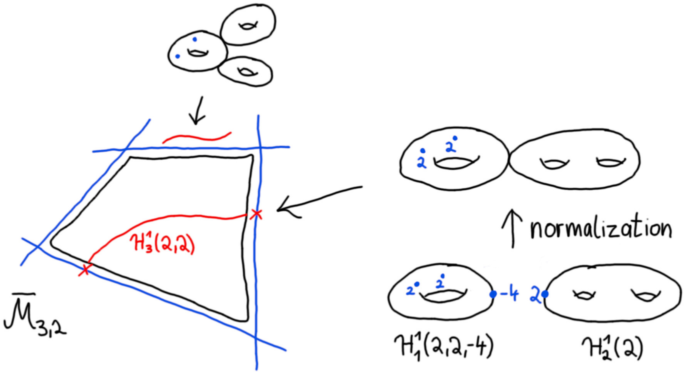

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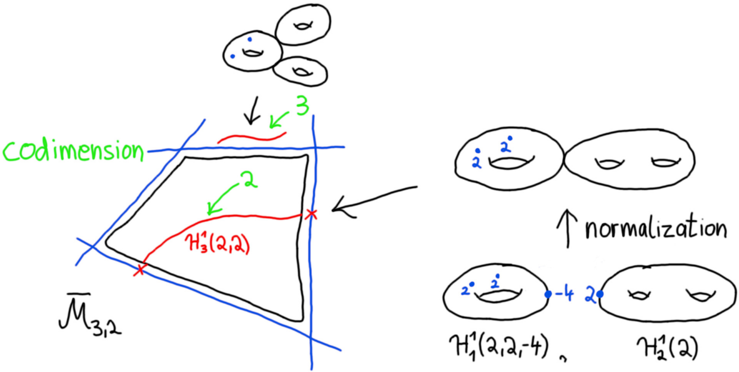

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Note

We have $\mathcal{H}_g^1(\mu') \subset \mathcal{H}_g^k(\mu)$ since

$$\omega_C \cong \mathcal{O}_C(\sum_i \frac{m_i}{k} p_i) \implies \omega_C^{\otimes k} \cong \mathcal{O}_C(\sum_i m_i p_i).$$

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- explicit list of components $[Z]$, each parametrized by products of $\overline{\mathcal{H}}_{g_j}^{k_j}(\mu_j)$

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Application : Recursion for $[\overline{\mathcal{H}}_g^k(\mu)]$

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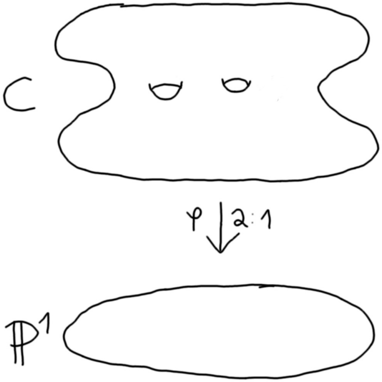

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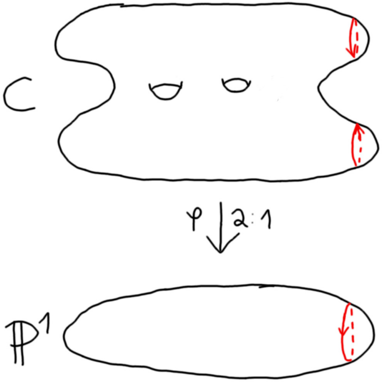

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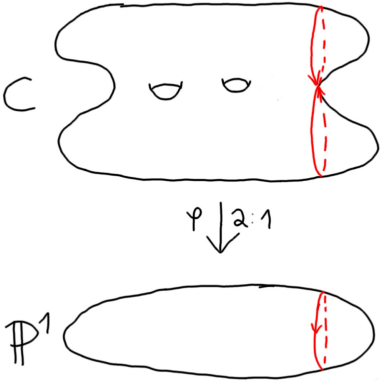

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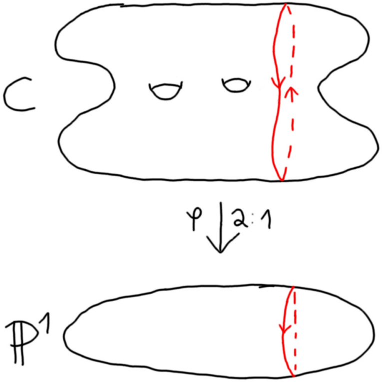

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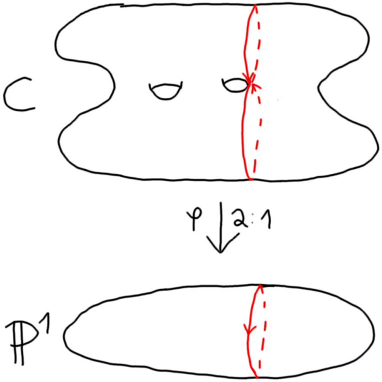

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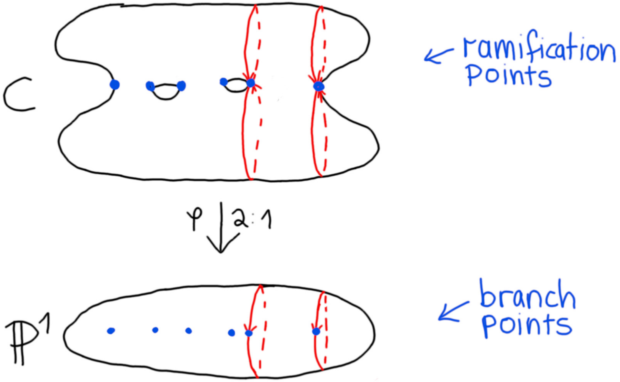

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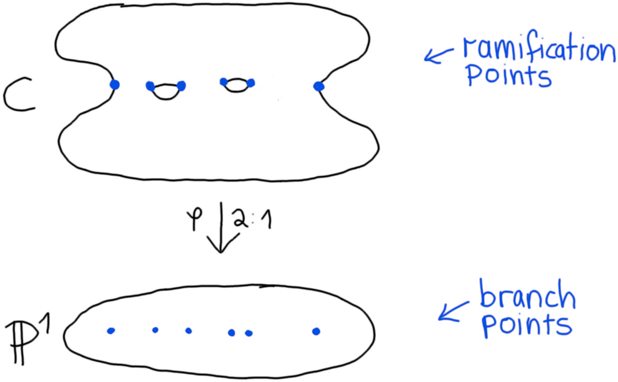

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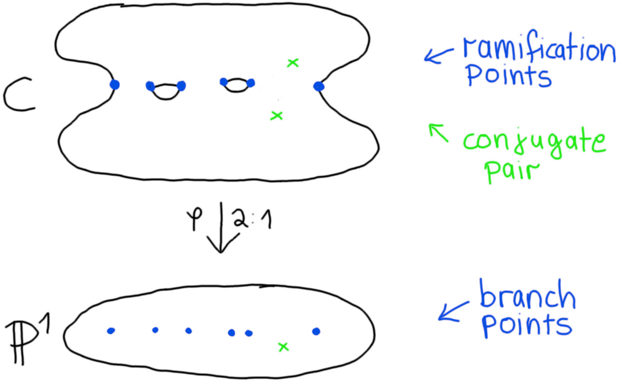

Loci of hyperelliptic and bielliptic curves

Definition

Let $g, n, m \geq 0$ be integers with $0 \leq n \leq 2g + 2$. Define

$$\mathrm{Hyp}_{g,n,2m} = \left\{ (C, (p_i)_{i=1}^n, (q_j, q_j')_{j=1}^m) : \begin{array}{c} C \text{ hyperelliptic} \\ \text{ram. points } p_i, \\ \text{conj. pairs } q_j, q_j' \end{array} \right\} \subset \mathcal{M}_{g,n+2m}.$$

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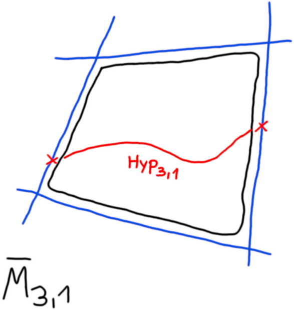
Compactification via admissible covers



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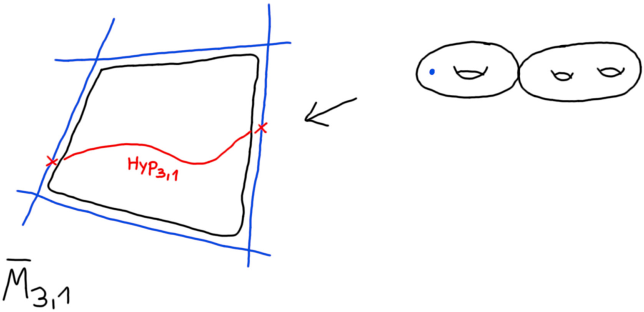

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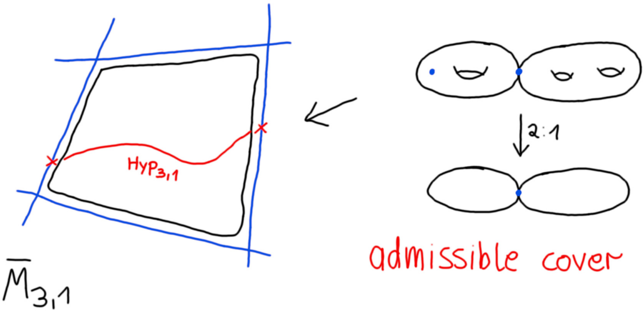

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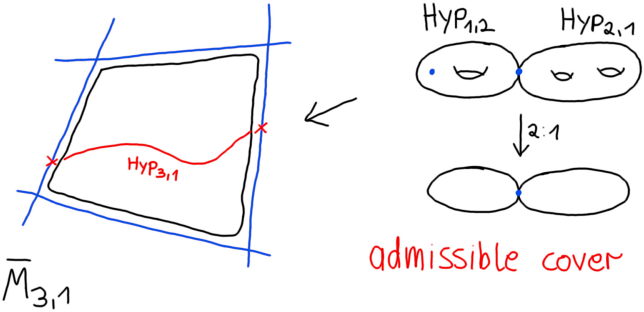

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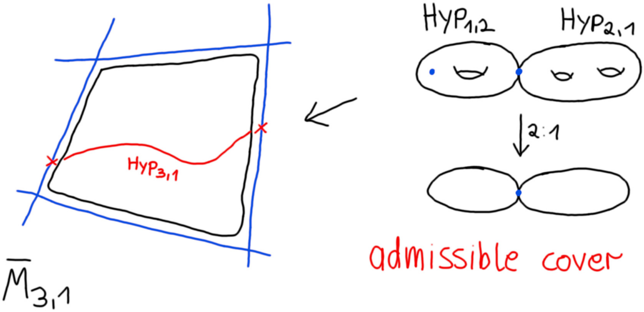

Goal

Study admissible cover cycles like $[\overline{\mathrm{Hyp}}_{g,n,2m}]$ and $[\overline{\mathrm{B}}_{g,n,2m}] \in H^*(\overline{\mathcal{M}}_{g,n+2m})$.

Admissible cover cycles

Theorem (Faber-Pandharipande 2005)

The fundamental class $[\overline{\mathrm{Hyp}}_{g,n,2m}] \in H^{2g+2n+2m-4}(\overline{\mathcal{M}}_{g,n+2m})$ lies in the tautological ring $RH^{2g+2n+2m-4}(\overline{\mathcal{M}}_{g,n+2m})$.

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Theorem

The fundamental class $[\overline{\mathrm{B}}_{g,n,2m}] \in H^{2g+2n+2m-2}(\overline{\mathcal{M}}_{g,n+2m})$ does not lie in the tautological ring $RH^{2g+2n+2m-2}(\overline{\mathcal{M}}_{g,n})$ for

- $(g, n, m) = (2, 0, 10)$ (Graber-Pandharipande 2003)
- $g \geq 2$ and $g + m \geq 12$ (van Zelm 2016)

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Note

For small (g, n, m) the cycle $[\overline{\mathrm{B}}_{g,n,2m}]$ is tautological, since $H^*(\overline{\mathcal{M}}_{g,n+2m}) = RH^*(\overline{\mathcal{M}}_{g,n+2m})$.

Strategy for computation

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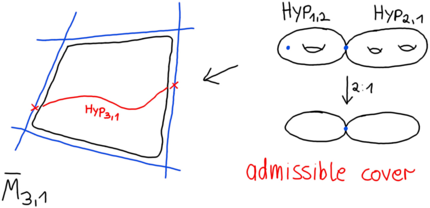

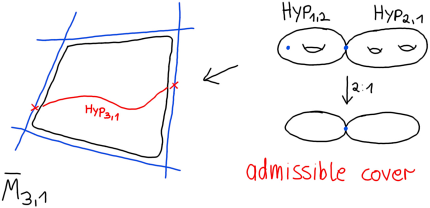

Lemma (Arbarello-Cornalba 1998)

For the inclusion $i : \partial\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ the pullback

$$i^* : H^k(\overline{\mathcal{M}}_{g,n}) \to H^k(\partial\overline{\mathcal{M}}_{g,n})$$

is injective for $k \leq d(g,n)$ with

$$d(g,n) = \begin{cases} n-4 & \text{if } g=0, \\ 2g-2 & \text{if } n=0, \\ 2g-3+n & \text{if } g>0, \\ & \quad n>0. \end{cases}$$

Computer package `admcycles`

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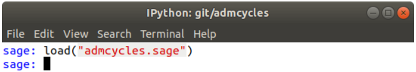

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Features

- computations with tautological classes (products and intersection numbers)
- verification of tautological relations
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- identification of admissible cover cycles in terms of tautological cycles

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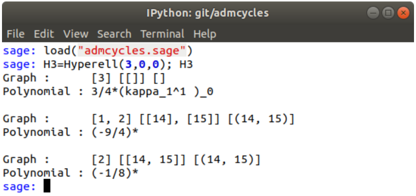

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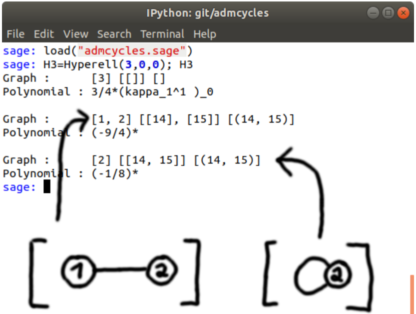

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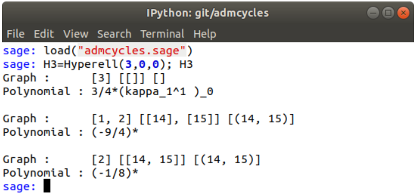

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- identification of admissible cover cycles in terms of tautological cycles

Computer package admcycles

```
IPython: git/admcycles
File Edit View Search Terminal Help
sage: load("admcycles.sage")
sage: H3=Hyperell(3,0,0); H3
Graph :      [3] [[]] []
Polynomial : 3/4*(kappa_1^1 )_0

Graph :      [1, 2] [[14], [15]] [(14, 15)]
Polynomial : (-9/4)*

Graph :      [2] [[14, 15]] [(14, 15)]
Polynomial : (-1/8)*
sage: g=3; n=0;
sage: (H3*kappaclass(1)^5).evaluate()
3197/8960
sage: 
```

Written in Sage (Python) with Jason van Zelm, Vincent Delecroix; based on earlier implementation by Pixton

Features

- computations with tautological classes (products and intersection numbers)
- verification of tautological relations
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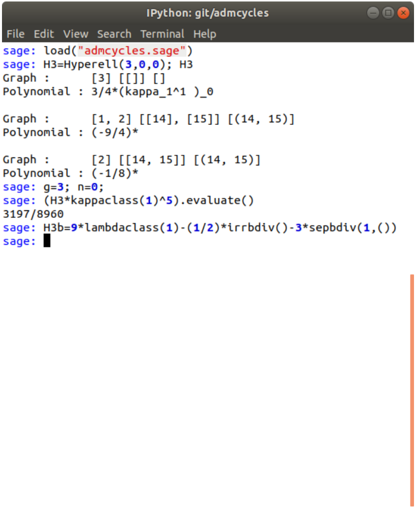

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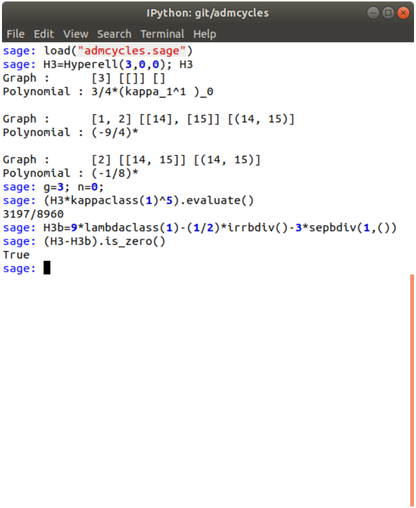

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sage: g=3; n=0;
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3197/8960
sage: H3b=9*lambdaclass(1)-(1/2)*irrbdiv()-3*sepbdiv(1,())
sage: (H3-H3b).is_zero()
True
sage:
```

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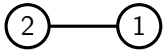
$[\overline{\mathrm{Hyp}_3}] = \frac{3}{4}\kappa_1 - \frac{9}{4}$ [] $- \frac{1}{8}$ []

(Harris-Mumford 1982)

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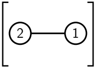
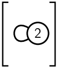
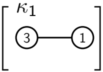
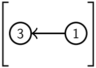
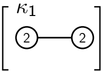
$[\overline{\mathrm{Hyp}_3}] = \quad \frac{3}{4}\kappa_1 \quad - \frac{9}{4}$ 

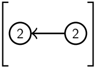
$-\frac{1}{8}$ 

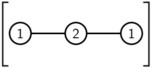
(Harris-Mumford 1982)

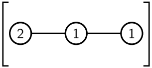
$[\overline{\mathrm{Hyp}_4}] = \quad \frac{17}{2}\kappa_2 \quad - \frac{17}{24}\kappa_1^2 \quad + \frac{7}{12}$ 

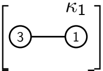
$-\frac{163}{24}$ 

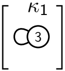
$+\frac{11}{12}$ 

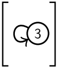
$-\frac{49}{8}$ 

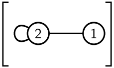
$+\frac{31}{24}$ 

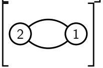
$+\frac{11}{12}$ 

$+\frac{163}{24}$ 

$+\frac{1}{12}$ 

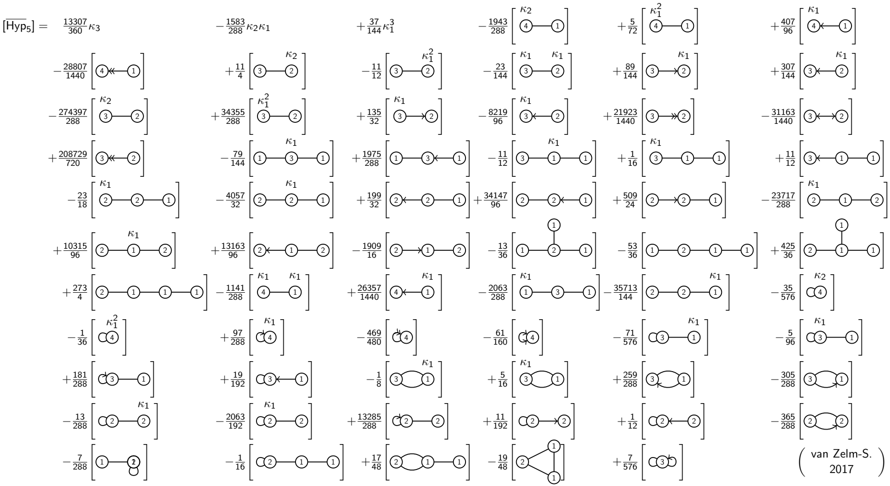
$-\frac{5}{8}$ 

$+\frac{1}{12}$ 

$-\frac{3}{8}$ 

(Faber-Pandharipande 2005)

Results



Results

$$[\overline{\mathrm{Hyp}_6}] =$$

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Other hyperelliptic and bielliptic cycles

Using `admcycles` one can compute the following cycles

Hyperelliptic cycles $[\overline{Hyp}_{g,n,2m}]$

g	0			1					2				3		4		5		6
n	0	1	2	0	1	2	3	4	0	1	2	3	0	1	0	1	0	1	0
m	4	3	2	2	2	1	1	0	2	0	0	0	1	0	0	0	0	0	0

(Faber-Pagani '12) (Harris-Mumford '82) (Faber-Pandharipande '05)

(Vermeire '02) (Cavalieri-Tarasca '17: n=1, ...,5) (Chen-Tarasca '15)

Bielliptic cycles $[\overline{B}_{g,n,2m}]$

g	1	2			3		4
n	0	0	1	2	0	1	0
m	2	1	0	0	0	0	0

(Faber '96) (Faber-Pagani '12)

Summary

- $\overline{\mathcal{M}}_{g,n}$ smooth, compact moduli space
- $RH^*(\overline{\mathcal{M}}_{g,n}) \subset H^*(\overline{\mathcal{M}}_{g,n})$ tautological ring, explicit generators $[\Gamma, \alpha]$
- $\widetilde{\mathcal{H}}_g^k(\mu)$ moduli space of twisted k-differentials
  - generalizes condition $\omega_C^{\otimes k} \cong \mathcal{O}_C(\sum_i m_i p_i)$
  - Theorem about dimension of the components of $\widetilde{\mathcal{H}}_g^k(\mu)$
  - Conjecture about formula for weighted fundamental class of $\widetilde{\mathcal{H}}_g^k(\mu)$ as tautological classes
- $\overline{Hyp}_{g,n,2m}$, example of admissible cover cycle
  - generalizes condition C hyperelliptic with ramification points p_i, conjugate pairs q_j, q_j'
  - Algorithm for restriction of $[\overline{Hyp}_{g,n,2m}]$ to boundary of $\overline{\mathcal{M}}_{g,n}$
  - Computation of new examples of formulas for $[\overline{Hyp}_{g,n,2m}]$

Crucial ingredient: recursive boundary structure of moduli spaces

Thank you for your attention!