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Truncated fusion rules for supergroups

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A new type of symmetry - supersymmetry - was introduced in physics in the 70's. The mathematical foundations of this theory involve the representation theory of the associated symmetry groups, the so-called supergroups. Here we would like to understand the fusion rules. In physical terms this is the question what happens during the fusion of two physical systems and how the new system is built from more fundamental building blocks. The answer is largely unknown, but one can get approximate answers in some cases.

1 Symmetries and groups

The concept of symmetry is one of the foundational principles in Mathematics and Physics. A symmetry of a system is an invariance under a transformation of the system. A sphere in three dimensional space looks the same (is invariant) after a rotation by an arbitrary angle, hence it is symmetric with respect to rotations. The set $\{1, 2, \ldots, n\}$ is symmetric with regards to arbitrary permutations of the numbers $1, 2, \ldots, n$ (since a set does not depend on the

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order of its elements). The transformations of the system which preserve partial configurations form the associated *symmetry group*. In the case of the sphere this is the *special orthogonal group* $SO(3, \mathbb{R})$, in the case of the set $\{1, \ldots, n\}$ this is the symmetric group S_n. Symmetries need not be geometric in nature as the permutation example shows. Other such examples arise in physics: Particles can have an inner symmetry called *Spin*. In the 70's physicists described a new conjectural form of symmetry: *supersymmetry*. Elementary particles can be divided into two families (according to their spin) called bosons (like the photon) and fermions (like the electron). The supersymmetry is a symmetry that can transform bosons and fermions into each other. Laying the mathematical foundations of supersymmetry has been an ongoing process since then. A key part of this is the representation theory of *supergroups*. It is in this area that we try to understand a mathematical problem that would physically correspond to the fusion of two physical systems.

Let us step back for a moment and return to the basic notations of a group. If we look at the set of rotations, it has a few remarkable properties:

1. If we take two rotations $\varphi_1, \ \varphi_2$, we get another rotation $\varphi_1 + \varphi_2$ by doing the rotations consecutively.
2. We can rotate by 0 degrees; this rotation leaves all points fixed.
3. For any rotation with angle φ there is an inverse rotation, namely we rotate by $-\varphi$. If we first do one and then the other, we rotate by 0-degrees.

In abstract terms we have a set (the rotations) in which any two elements can be composed to yield another element from this set, and there is a neutral element with respect to this composition (the rotation by 0-degrees) and an inverse (the rotation by the angle $-\varphi$). A set with such a composition is called a *group*. Groups abound in mathematics; obvious examples are sets of numbers: the real numbers $\mathbb{R}$ form a group with the usual addition: $a + b$ is another real number, the neutral element is 0 and the inverse to a is $-a$. Yet another example is the symmetric group mentioned above.

2 Representations of groups

Representation theory studies groups - or other similar structures - by representing them as linear transformations on vector spaces.

The prototypical example of a vector space is $\mathbb{R}^3$ - 3-dimensional space, or its generalization $\mathbb{R}^n$, n-dimensional space. An element in $\mathbb{R}^3$ is given by a 3-tuple (x, y, z) or (x_1, x_2, x_3) where x_1, x_2 and x_3 are real numbers.

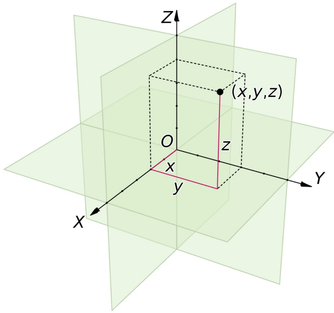

Figure 1: A point in $\mathbb{R}^3$

It is common to call these tuples vectors. But $\mathbb{R}^3$ is not just a set of vectors: We can add two tuples

$$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$$

and we can multiply them by real numbers

$$a \cdot (x_1, x_2, x_3) = (ax_1, ax_2, ax_3) \text{ for } a \in \mathbb{R}$$

and there is *zero vector*

$$(0, 0, 0) + (x_1, x_2, x_3) = (x_1, x_2, x_3).$$

Such a structure is called a *vector space* in mathematics. The vector space $\mathbb{R}^3$ is specified by three coordinates, but there are vector spaces which cannot be described by finitely many coordinates, therefore called infinite dimensional.

We now look at *linear maps* $\varphi : V \to V$ of a vector space V, the ones that are compatible with the addition and scalar operation

$$\varphi(ax + by) = a\varphi(x) + b\varphi(y),\ \ a, b \in \mathbb{R},\ \ x, y \in V.$$

A *representation* of a group G then assigns to every element $g \in G$ a linear map $\varphi_g : V \to V$. We can also write this as $G \to End(V)$, where $End(V)$ is the set of linear maps of V to itself; and we often say that G acts on V (via linear maps). A representation of G on a vector space allows one to study G by looking at the simplified picture of the associated linear transformations. By doing so we loose information about the group. We should therefore understand the whole collection of representations.

3 A quick primer on representation theory

The study of representations of groups (or similar algebraic structures) is called representation theory. It is a vast theory with many different flavours, but at the heart of the matter there are some simple questions such as: How many representations are there? Can one classify them? In this generality there is no hope to answer them. A first very harsh restriction might be to look at finite dimensional vector spaces. In a second step one should not hope to get uniform answers for all groups, but restrict to particular groups of interest (such as S_n or $SO(n,\mathbb{R})$). In a third step the problem of describing all finite dimensional representations should be reduced to some basic building blocks. Like all matter is composed of atoms, and all atoms ultimately of elementary particles, each representation should be built from some elementary or fundamental representations. It would then be enough to describe these elementary representations and furthermore all possible ways one can build other representations with them. These elementary building blocks are called *irreducible representations*. A representation is called irreducible if it does not contain any other (non-trivial) representation, i.e. there is no subvector space of V which is stable under the action of G.

Let us look at an example, the action of the symmetric group S_n on $\mathbb{R}^n$. An element in $\mathbb{R}^n$ is given by a tuple $(x_1, \ldots, x_n)$ of real numbers x_i. The permutation group acts on the set of such tuples by permuting the entries. If σ is the permutation in S_7 that swaps the 2-nd and 5-th entry, then σ sends $(x_1, x_2, x_3, x_4, , x_5, x_6, x_7)$ to the tuple $(x_1, x_5, x_3, x_4, x_2, x_6, x_7)$. In this way we can specify for each element in S_n a linear map from $\mathbb{R}^n \to \mathbb{R}^n$ which means that we have defined a representation of S_n on $\mathbb{R}^n$. But this representation is not irreducible! Indeed look at the vectors in $\mathbb{R}^n$ of the form $(x, x, \ldots, x)$ where x is an arbitrary real number. The space of such tuples is a 1-dimensional subspace of $\mathbb{R}^n$ (since an arbitrary element is specified by a single number x). This subspace is invariant under S_n: S_n sends $(x, \ldots, x)$ to itself and preserves this subspace. We have hence found a subrepresentation in $\mathbb{R}^n$ (which is therefore not irreducible). We identify this subrepresentation with $\mathbb{R}$ via the map $x \mapsto (x, x, \ldots, x)$. Conversely the set of tuples $(x_1, \ldots, x_n)$ for which not all x_i are the same is invariant under the S_n-action. Hence this is also a subrepresentation, this time of dimension $n-1$ ($\mathbb{R}^n$ has dimension n and the dimensions of the two subrepresentations must add up to n). This representation is sometimes called the standard representation st of S_n. Clearly the underlying sets of the two subrepresentations are disjoint and any vector in $\mathbb{R}^n$ is in one of them. We write this as

$$\mathbb{R}^n \cong \mathbb{R} \oplus st \quad \text{(as representations of } S_n\text{)}.$$

It can be checked that st is irreducible, hence the representation of S_n on

$\mathbb{R}^n$ is the direct sum of two irreducible representations. In fact a theorem of Maschke asserts that every finite dimensional representation of a finite grup on a real or complex vector space can be written as a direct sum of irreducible representations. This theorem is no longer true if one works with infinite groups or infinite dimensional vector spaces.

The representation theory of the symmetric groups is an old classical subject. Much of it goes back to the work of Isaai Schur and Georg Frobenius more than a hundred years ago.

Figure 2: Isaai Schur (left) and Georg Frobenius (right)

But even here one encounters elementary, yet unsolved questions. One such question concerns the *fusion rules*. Given two representations V, W of a group G (this means that V and W are vector spaces on which G acts), we can use these to construct new representations. One such construction is the *direct sum* $\oplus$ used above. Writing $V \oplus W$ means looking at the tuples (v, w) where $v \in V,\ w \in W$. These tuples are closed under addition $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$ and multiplication by a real number $a \cdot (v, w) = (av, aw)$ and hence form a vector space of dimension $\dim(V) + \dim(W)$. The vector space $V \oplus W$ is a representation of G if we define $g \cdot (v, w) = (g \cdot v, g \cdot w)$. One example of such a construction is $\mathbb{R}^2$: As a vector space this is simply $\mathbb{R} \oplus \mathbb{R}$. Yet another way to build a new representation is the *tensor product* $V \otimes W$. This is again a new vector space, but this time of dimension $\dim(V) \cdot \dim(W)$. It contains elements $v \otimes w$ for $v \in V,\ w \in W$, but also more complicated expressions (sometimes called entangled states) of the form $\sum_{i,j} v_i \otimes w_j$ for v_i some vectors in V and some vectors w_j in W.

In general any representation can be written as a direct sum of *indecomposable* representations. These are representations which can not be decomposed any further (though they might not be irreducible). This applies therefore also to the tensor product $V \otimes W$ and we can write

$$V \otimes W = c_{V,W}^{I_1} I_1 \oplus \ldots \oplus c_{V,W}^{I_n} I_n$$

for certain indecomposable representations I_i of G and certain multiplicities $c_{V,W}^{I_i} \in \mathbb{N}$ (which count how often I_i turns up in this decomposition). A rule which tells us what these coefficients are and how they are computed (and therefore how the tensor product $V \otimes W$ decomposes) is called a *fusion rule*.

In the situation of the symmetric group, Maschke's theorem says that any finite dimensional representation can be written as a direct sum of irreducible representations. In this case the coefficients $c_{V,W}^{I_\nu}$ have a special name: they are called *Kronecker coefficients*. While the finite dimensional representation theory of the symmetric group over the real or complex numbers is classical and quite elementary, we don't know any good description of these coefficients yet! We would like to have a closed combinatorial expression of the $c_{V,W}^{I_\nu}$, but this unknown. The lesson is that finding the fusion rules is going to be very hard in general if we already fail for such a well-studied example as S_n.

4 Continuous symmetries and Lie groups

It is important to take into account that there are many types of groups, and hence we cannot expect a single theory that describes all possible representations of groups. For example, groups could describe continuous symmetries (such as $SO(3, \mathbb{R})$) or discrete symmetries (such as S_n). Continuous symmetries lead to the theory of *Lie groups*, named after the Norwegian mathematician Sophus Lie. An analysis of their representations requires other methods than the study of discrete groups such as S_n.

The most important Lie group is $GL(n, \mathbb{C})$, the group of invertible linear maps from $\mathbb{C}^n \to \mathbb{C}^n$, or its real analog $GL(n, \mathbb{R})$. Many other important Lie groups occur naturally as subgroups. For this it is convenient to identify linear maps $\mathbb{C}^n \to \mathbb{C}^n$ or $\mathbb{R}^n \to \mathbb{R}^n$ by matrices in the following way. Each vector of either $\mathbb{R}^n$ or $\mathbb{C}^n$ can be written as a linear combination of the *standard basis*

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \ldots \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \ldots \\ 0 \end{pmatrix}, \ldots, e_n = \begin{pmatrix} 0 \\ 0 \\ \ldots \\ 0 \\ 1 \end{pmatrix}.$$

If v has coordinates

$$v = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \dots \\ \lambda_n \end{pmatrix}$$

with $\lambda_i \in \mathbb{C}$ or $\mathbb{R}$, the vector v is given by

$$v = \lambda_1 e_1 + \lambda_2 e_2 + \ldots + \lambda_n e_n.$$

For any linear map $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ we get

$$\varphi(v) = \varphi(\sum_{i=1}^{n} \lambda_i e_i) = \sum_{i=1}^{n} \lambda_i \varphi(e_i).$$

Therefore a linear map is completely determined by its values $\varphi(e_i)$ on the basis vectors ($i = 1, \ldots, n$). It is customary to collect this information into an $n \times n$-matrix defined by

$$M(\varphi) = \begin{pmatrix} \varphi(e_1) & \varphi(e_2) & \ldots & \varphi(e_n) \end{pmatrix}.$$

The collection of all possible $n \times n$-matrices (with n rows and n columns) with real or complex entries is denoted by $M_{n\times n}(\mathbb{R})$ or $M_{n\times n}(\mathbb{C})$. These matrices correspond 1:1 to linear maps from $\mathbb{R}^n \to \mathbb{R}^n$ or $\mathbb{C}^n \to \mathbb{C}^n$. The groups $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ can be identified with real and complex matrix with non-vanishing *determinant* (The determinant of a matrix is a real or complex number that measures whether the corresponding linear map is invertible).

One advantage of this viewpoint is that by representing elements of $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$ by matrices we immediately obtain a representation of $GL(n)$. The n-dimensional representation $\mathbb{R}^n$ or $\mathbb{C}^n$ defined by the action of matrices on elements of $\mathbb{R}^n$ and $\mathbb{C}^n$ is called the *standard representation.*

Another advantage one is that one can easily define some important subgroups of $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ now:

- The *special linear groups* $SL(n, \mathbb{R})$ and $SL(n, \mathbb{C})$ consisting of $n \times n$ matrices with determinant one and entries in $\mathbb{R}$ or $\mathbb{C}$.
- The real or complex *orthogonal* and *special orthogonal* groups, $O(n, \mathbb{R})$, $SO(n, \mathbb{R})$, $O(n, \mathbb{C})$, $SO(n, \mathbb{C})$ consisting of (real or complex) $n \times n$ matrices satisfying $R^T = R^{-1}$ (and also $det(R) = 1$ in the case of $SO(n)$). Here R^T is the transposed matrix, the matrix obtained by flipping all entries over the diagonal.
- The *unitary* groups and *special unitary* groups $U(n)$ and $SU(n)$ consisting of $n \times n$ complex matrices satisfying $R^* = R^{-1}$ (and $det(R) = 1$ in the case of $SU(n)$). Here R^* is the conjugate transpose, the matrix obtained by transposition and then applying complex conjugation to each entry.

- Other notable examples are the series of *symplectic* groups $Sp(2n, \mathbb{C})$ and several exceptional Lie groups.

As for finite groups one can now study (finite dimensional) representations of Lie groups. Here again we consider (linear) actions of G on a finite dimensional vector space such as $\mathbb{C}^n$, but we require that the map is *smooth*, taking into account the continuous symmetry. In this note we only look at the more restrictive class of *algebraic representations.*

The representation theory of Lie groups is an extremely rich subject that has connections to almost all areas of pure mathematics. In general, even finite dimensional representations over the complex numbers need not be completely reducible, but this is still true if one looks at algebraic representations of matrix groups such as $GL(n)$, $SO(n)$, $O(n)$, $Sp(2n)$ and $SU(n)$ (Weyl's complete reducibility theorem)(for simplicity we will only work over $\mathbb{C}$ and assume we are dealing with the complex groups from now on). Contrary to the finite group case there are always infinitely many irreducible representations, but they can often be classified. Coming back to our old problem, we can ask now how we can decompose the tensor product of two representations into a direct sum of irreducible or indecomposable representations.

For groups of continuous symmetries such as $SL(n)$, $GL(n)$, $SU(n)$, $SO(n)$ there are algorithmic descriptions of these fusion rules (the *Littlewood-Richardson rule* for $GL(n)$ and $SL(n)$ and variants of it for other classical groups). In fact Littlewood-Richardson "conjectured" an algorithm for the this problem in the $SL(n)$-case in 1934 which was finally proven in the 70's by Schuetzenberger and Thomas. In the words of Gordon James:

> *Unfortunately the Littlewood-Richardson rule is much harder to prove than was at first suspected. The author was once told that the Littlewood-Richardson rule helped to get men on the moon but was not proved until after they got there.*

Indeed as soon as the dimensions of the representations become large, their tensor product decomposes into zillions of summands with no obvious pattern.

5 Algebraic aspects of the standard model

Physicists are often not interested in a random group, they mostly need very special low rank Lie groups: $U(1)$ for *translation symmetry*, $SO(3, \mathbb{R})$ for *spatial rotations*, $SU(2)$ to describe *isospin* and a few others more. The most important groups are those that arise as *gauge groups* in gauge field theories, notably the *Standard Model*, a special kind of quantum field theory.

The symmetry group of the standard model and the classifaction of elementary particles is based on a feedback loop between symmetry considerations (i.e. the representation theory of possible groups) and empirical data on the other side. High energy collision experiments and the subsequent analysis of the data suggested conservation laws and symmetry constraints which in turn allowed the prediction of new particles that could ultimately be found in experiments.

Ultimately the goal is to describe all matter in terms of elementary particles and the interactions between them by fundamental forces. The Standard model achieves this excluding a description of gravity. The dynamics and kinetic of the theory is described by a *Lagrangian* $\mathcal{L}$. To get this Lagrangian one first postulates a set of symmetries and then tries to find the most general form a Lagrangian can have that satisfies these symmetries. Like in all relativistic field theories, the Lagrangian has to observe so-called *global Poincaré symmetry*, that is invariance under Lorentz transformations and translations in Minkowski space. Additionally the Lagrangian has an internal symmetry (*local gauge symmetry*) with respect to the gauge group $G = U(1) \times SU(2) \times SU(3)$. The three symmetry groups belong to the three interactions, electromagnetism and weak and strong force that the Standard Model incorporates. The fields in the Standard Model are then build from the fields of the three different symmetry groups (which we should think of as taking values in irreducible representations).

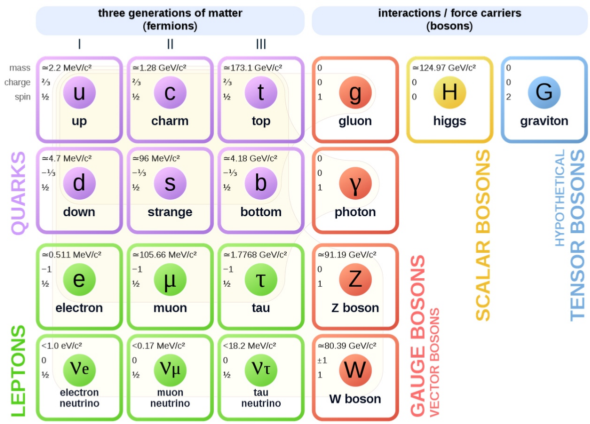

Figure 3: The elementary particles in the standard model

But how do we get the irreducible representations of a product group like $U(1) \times SU(2) \times SU(3)$ if we know the irreducible representations of each factor? The answer is fortunately simple: We take the *external tensor product* $\boxtimes$ of the irreducible representations of each factor. So knowing the irreducible representations of $U(1)$, $SU(2)$ and $SU(3)$ is enough to get the irreducible representations of the product. In down to earth terms, if U, V, W are irreducible representations of $U(1)$ respectively $SU(2)$ respectively $SU(3)$, then $U \boxtimes V \boxtimes W$ is an irreducible representation of G, and every irreducible representation of G is of this form.

In the standard model the elementary particles (like the electron or the quarks) sit inside an inside an irreducible representation of $G = SU(3) \times SU(2) \times U(1)$. The different elementary particles span different irreducible representations of G. An example is given by the up-quark. It comes in three different polarizations (sometimes called red, green, blue). Together they combine to the standard representation $\mathbb{C}^3$ of $SU(3)$ (when ignoring the $SU(2) \times U(1)$-part).

This description shows part of the algebraic structure of the Standard Model. However the Standard Model is much more, it doesn't just classify elementary particles by representations. As any quantum field theory it describes the dynamics of the system: How do particles move and interact? The mathematical and physical descriptions of this are far beyond this little paper. For our purpose let us note that here the Lagrangian comes into play. The evolution of a physical state in the Standard Model is given by the so-called *path integral* (or *Feynman integral*) formalism. These Feynman integrals are not mathematically rigorously defined, but they have been used by physicists since decades to calculate the effect of collisions and other physical effects with very high precision.

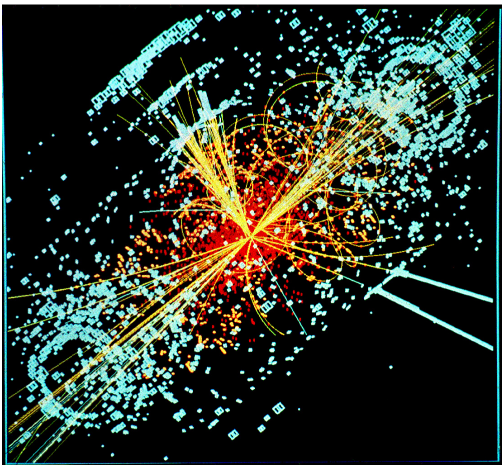

Figure 4: Data from a particle collision

Whatever the methods of computation are, if we want to evaluate a Feynman integral that describes the collision of two particles, the result depends on the fusion rules between the corresponding irreducible representations. While there are many open questions about these Feynman integrals, the part that involves the fusion rules is a classical piece of mathematics and well-understood. The situation changes considerably if one tries to replace the Standard Model and its gauge group by a more complicated (?) theory based on the notion of *supersymmetry*.

6 Super structures

While the Standard Model had tremendous success in unifying electromagnetism and the weak and strong force and agrees with experimental results, it has several shortcomings which lead physicists to search for alternatives. In particular one would like to have a supersymmetric extension of the Standard Model. So far no experimental evidence for such an extension was found at the LHC or other colliders, and hence the concept of supersymmetry remains in limbo despite its theoretical advantages.

Mathematically the passage to the supersymmetric extension involves replacing the Lie group with a Lie supergroup, a $\mathbb{Z}_2$-graded generalization of the former. This means that the new algebraic structure has an *even part* (the *bosonic part*) and an *odd part* (the *fermionic part*). Similarly the vector spaces on which these groups or algebras act are replaced by *super vector spaces*, vector spaces with an even and an odd part. The easiest example is simply

$$\mathbb{C}^{m|n} := \mathbb{C}^m \oplus \mathbb{C}^n,$$

a vector space with two parts, one seen as even ($\mathbb{C}^m$), the other one as odd ($\mathbb{C}^n$). We can now look at all linear transformations of $\mathbb{C}^{m|n}$.

The $\mathbb{Z}_2$-grading allows to introduce a $\mathbb{Z}_2$-grading on the space of linear maps of ($\mathbb{C}^{m|n}$) to itself: a transformation $\mathbb{C}^{m|n} \to \mathbb{C}^{m|n}$ is even if it maps the even part to the even part and the odd part to the odd part. It is called odd, if it maps the even to odd part and vice versa. Hence the space of linear maps of $\mathbb{C}^{m|n}$ is itself a super vector space! Copying the definition of $GL(n,\mathbb{C})$ we define

$$GL(m|n,\mathbb{C}), \quad \textit{the General Linear Supergroup}$$

to be the group of invertible linear maps $\mathbb{C}^{m|n} \to \mathbb{C}^{m|n}$. The super world incorporates the classical case via $GL(m|0,\mathbb{C}) = GL(m,\mathbb{C})$. As in the classical world there are analogs of $SO(n)$ and $Sp(2n)$ (the *orthosymplectic supergroups*) but also new types of groups which have no classical counterpart.

Similarly to the classical theory of Lie groups, one can now ask: What are the irreducible representations? What are their dimensions? Can every representation be written as a direct sum of irreducible representations? These and further questions have been investigated since the foundational work of Kac [Kac78] in the 70's; and the study of (algebraic) representations of Lie supergroups has now become a thriving area in pure mathematics with connections to many other fields such as algebraic geometry, quantum topology and even analytic number theory.

7 Fusion rules for supergroups and truncations

One major difference is that Weyl's theorem fails in the super world: Not all finite dimensional representations can be written as a sum of irreducible representations. This means that we will encounter representations which have many subrepresentations, but there is no way to split them into a direct sum of two other representations (called indecomposable). The occurence of such representations renders many tools from classical Lie theory useless; and the questions posed above have much more complicated answers or are even unknown. In the beginning there is a fortunate surprise: It is quite easy to parametrize the irreducible representations for many Lie supergroups.

In the $GL(m|n)$-case irreducible representations are parametrized by tuples

$$\lambda = (\lambda_1, \ldots, \lambda_m \mid \lambda_{m+1}, \ldots, \lambda_{m+n}) \in \mathbb{Z}^{m|n}$$

where $\lambda_1 \geq \ldots \geq \lambda_m$ and $\lambda_{m+1} \geq \ldots \geq \lambda_{m+n}$. We simply denote the corresponding irreducible representation by $L(\lambda)$. Actually for each λ there is a second irreducible representation, namely the parity shift $\Pi L(\lambda)$ where we swap the even and the odd part of the underlying super vector space.

Many of questions that one can ask about these $L(\lambda)$'s turn out to be very hard: What is the dimension of $L(\lambda)$? What is its superdimension? In what ways can we combine such $L(\lambda)$ to form these big complicated indecomposable modules? And, most importantly for this article, what are the fusion rules for $L(\lambda) \otimes L(\mu)$? Some of these answers have been answered in the last 10 - 20 years [Ser10] [BS12] [HW21] but there are still open questions, and the situation is even more complicated for other Lie supergroups.

Quite generally the fusion rules are known for the case $GL(m|1)$, but beyond that the decomposition $L(\lambda) \otimes L(\mu)$ is only known for special weights λ, μ. It was suggested in [Hei15] and carried out in [HW18] [HW22] that one should instead look at *truncated fusion rules*: A super vector vector space might have an even part with the same dimension as the odd part. The difference of the dimensions $\dim(V_0) - \dim(V_1)$ is called the *superdimension* $\text{sdim}(V)$. Consider for example the standard representation $\mathbb{C}^{m|n}$ of $GL(m|n)$: Its superdimension is $m - n$.

We then look at tensor products of irreducible representations of non-vanishing superdimension and calculate the decomposition *modulo superdi-*

mension 0: In the formula $L(\lambda) \otimes L(\mu) = I_1 \oplus I_2 \oplus \ldots \oplus I_n$ we disregard all indecomposable summands such that $\mathrm{sdim}(I_\nu) = 0$. While the actual fusion rules are unknown, it turns out that one can determine these truncated fusion rules in almost all cases!

Our main result [HW18] essentially says that the fusion rule describing this decomposition is the same as the one for a classical groups such as $Sp(2n)$, $SU(n)$ or $SL(n)$ and so on!

The key point is that one can attach to an irreducible representation $L(\lambda)$ of $GL(m|n)$ a group H_λ (which is now not a supergroup, but a classical group like those above) and an irreducible representation $V(\lambda)$ of H_λ such that $L(\lambda) \otimes L(\mu)$ decomposes exactly as $V(\lambda) \otimes V(\mu)$ decomposes. The groups H_λ and the representations $V(\lambda)$ are completely explicitely given.

How does this work in practice? Suppose we want to decompose the tensor product of $L(2,1,0|0,-1,-2)$ of $GL(3|3)$ with itself. (we use the abbreviation $[2,1,0]$). Our main theorems tell us that the corresponding pair is $(Sp(6), L(1,0,0))$. Here $L(1,0,0)$ is actually just the standard representation of $Sp(6)$: $\mathbb{C}^6$ with the action of $Sp(6)$ given by matrix multiplication. The classical tensor product rules for $Sp(6)$ tell us that $L(1,0,0)^{\otimes 2}$ decomposes as

$$L(1,0,0)^{\otimes 2} \cong L(0,0,0) \oplus L(2,0,0) \oplus L(1,1,0).$$

To each of the three summands corresponds an indecomposable representation of $\mathfrak{gl}(3|3)$ whose superdimension agrees with the dimension of the $Sp(6)$-representations. Hence

$$[2,1,0] \otimes [2,1,0] \cong I_1 \oplus I_2 \oplus I_3$$

up to contributions of superdimension 0. But now we can iterate this further and further and apply this to tensor products between the indecomposable summands that appear in this way. Let's say I_3 corresponds to $L(1,1,0)$. In order to compute $I_3 \otimes I_3$ up to superdimension 0, we can look at $L(1,1,0)^{\otimes 2}$ and match the resulting summands with indecomposable summands in $I_3^{\otimes 2}$ and so forth.

If we would have taken an irreducible representation of $GL(9|3)$ instead, the associated group H_λ would have been $GL(6) \times H_{\lambda'}$ for a group $H_{\lambda'}$ which can be calculated in the $GL(3|3)$-case. Then our tensor product decomposition would have been a mix of the Littlewood-Richardson rule for $GL(6)$ and the rule for $H_{\lambda'}$ (e.g. $Sp(6)$).

Alas there is a small caveat: For some special weights we cannot determine H_λ completely. In these cases we have two candidates for H_λ and cannot decide which of the two is actually the right one.

8 Back to physics?

While these result can be seen as a first step to obtain general fusion rules, one may wonder or dream whether there is more to it, whether maybe the truncation at sdim $= 0$ actually has a physical meaning?

Let us suppose we have a supersymmetric extension of the Standard Model with a super gauge group G. For supersymmetric fields ψ on super Minkowski space M (a combination of 3-dimensional euclidian space and time to a 4-dimensional superspace) with values in a finite dimensional representation V of G, the computation of the Feynman integrals require the analysis of higher tensor products $V^{\otimes r} \otimes (V^\vee)^{\otimes s}$ (here $V^\vee$ is the dual representation). These are approximated by our truncated fusion rules.

A particular important case is the one of $GL(4|4)$. Its importance arises from the connection to the *conformal group* of Minkowski space which consists of those transformations that preserve angles. Every Lie group has an associated *infinitesal symmetry group* attached to it (its *Lie algebra*) which controls to a good extent its structure and representation theory. In the case of the conformal group this Lie algebra is denoted $\mathfrak{so}(2,4)$; it lives naturally over $\mathbb{R}$ but can be extended to $\mathbb{C}$. Its complexification is isomorphic to $\mathfrak{sl}(4,\mathbb{C})$, the Lie algebra of $GL(4,\mathbb{C})$ which gives a correspondence between irreducible representations of $GL(4,\mathbb{C})$ and $\mathfrak{so}(2,4)$. Generalizing to the superworld, the groups $GL(4|N,\mathbb{C})$ and their representation are the appropriate setup to study superconformal extensions of the conformal group. Here is a list of small cases for $GL(4|4)$ (where Sp^c stands for the compact symplectic group):

- $[3,2,1,0]$, sdim $= 24$, $H_\lambda = SO(24)$ (conjecturally).
- $[3,2,0,0]$, sdim $= 12$, $H_\lambda = SU(12)$.
- $[3,1,1,0]$, sdim $= 12$, $H_\lambda = Sp^c(12)$.
- $[3,1,0,0]$, sdim $= 8$, $H_\lambda = SU(8)$.
- $[3,0,0,0]$, sdim $= 4$, $H_\lambda = SU(4)$.
- $[2,2,1,0]$, sdim $= 12$, $H_\lambda = SU(12)$.
- $[2,2,0,0]$, sdim $= 6$, $H_\lambda = SO(6)$.
- $[2,1,1,0]$, sdim $= 8$, $H_\lambda = SU(8)$.
- $[2,1,0,0]$, sdim $= 6$, $H_\lambda = Sp^c(6)$.
- $[2,0,0,0]$, sdim $= 3$, $H_\lambda = SU(3)$.
- $[1,1,1,0]$, sdim $= 4$, $H_\lambda = SU(4)$.
- $[1,1,0,0]$, sdim $= 3$, $H_\lambda = SU(3)$.
- $[1,0,0,0]$, sdim $= 2$, $H_\lambda = SU(2)$.
- $[1,1,1,1]$, sdim $= 1$, $H_\lambda = U(1)$.

In fact in the first example we cannot rule out the possibility that $H_\lambda = SU(12) \rtimes \mathbb{Z}_2$. The reader will observe that the smallest arising groups are

$U(1)$, $SU(2)$ and $SU(3)$. One may ask whether the appearence of the groups $U(1), SU(2), SU(3)$ here is a mere accident, or whether there does exist some connection with the symmetry groups arising in the standard model of elementary particle physics?

One special feature of supersymmetric field theories are cancellations due to the appearances of minus signs and the existence of superpartners. These cancellations are responsible for many of the more amenable features of such theories compared to the standard model. If in such a theory, for some mysterious physical reasons, the tensor product contributions to the Feynman integrals from direct summands of $V^{\otimes r} \otimes (V^{\vee})^{\otimes s}$ of superdimension zero would be relatively small in a certain energy range due to supersymmetry cancellations, then to first order they would be negligible. Hence a physical observer might come up with the impression that the underlying rules of symmetry are imposed by the invariant theory of the groups H_λ; and the groups H_λ or their product would appear as an internal symmetry group of the theory in an approximate sense. Of course this is highly speculative. Whether there exists any supersymmetry in nature at all, and whether our H_λ appear as approximative symmetry groups of such a theory, will probably take many years to uncover.

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